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MODERN SOLID GEOMETRY

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Ptolemy the First asked Euclid if he, the king, might not acquire a knowledge of geometry without studying the Elements. Euclid answered, "There is no royal road to learning." And this dictum has been accepted ever since by educationists, if not by kings. There is indeed no royal road to geometry. But, when the guidance is good, travel along the common road can be made an adventure filled with interest. A student learns most readily about that in which he is most interested, and his interest may be effectively invoked by offering challenges to his curiosity, imagination, and ingenuity. That idea underlies this new text on solid geometry. The authors have endeavored to present, not a royal road to geometry, but a road along which the obstacles shall serve as incentives, and at the end of which the student shall be rewarded with the satisfaction of having discovered for himself the great basal truths of geometry.

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PREFACE

TEACHERS of solid geometry appear to be in agreement that the primary purposes to be achieved through the study of solid geometry are (1) the *discovery* of geometrical truths of three-dimensional space and their *establishment* by *logical* methods, and (2) the development of computational skill in the mensuration of common geometrical solids.

Obviously a pupil cannot discover the truths of solid geometry or establish them by logic unless he analyzes the relationships of points, lines, and planes in space and is responsive to the same niceties of logic that are exercised in plane geometry; that is, geometry should be understood rather than memorized for reproduction.

Accordingly the authors have given careful attention to training the pupil in the analysis of three-dimensional space. The pupil learns to visualize a plane perpendicular to a line, a plane perpendicular to a plane, etc., thus establishing the basis for understanding the postulates and definitions of solid geometry.

For the development of computational skill, the authors have included a large number of carefully graded practical applications involving both arithmetical and algebraical computation.

It goes without saying that the recommendations of the National Committee on Mathematical Requirements and of the College Entrance Examination Board have been fully met in this textbook.

As in *Modern Plane Geometry*, the pupil is encouraged to attempt the formulation of proofs before reading them, as an exercise in independent thinking and resourcefulness. It has been found, moreover, that when a pupil has first

tried to formulate the proof himself, he understands more easily the proof given in the book.

The question form of exercise used in *Modern Plane Geometry* is used in this book. This device also calls for original thinking on the part of the pupil and encourages a healthy spirit of argument in the classroom. It is primarily through this question form of exercise that opportunity is given for the discovery of geometrical truths.

The instructional tests for each of the three books enable the pupil to measure his power in solid geometry.

The authors wish to express their indebtedness to Professor E. A. Bond of the State Teachers College at Bellingham, Washington, for invaluable assistance in the preparation of this book.

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REFERENCES TO PLANE GEOMETRY

AXIOMS

1. If equals are added to equals, the sums are equal.
2. If equals are subtracted from equals, the remainders are equal.
3. If equals are multiplied by equals, the products are equal.
4. If equals are divided by equals, the quotients are equal.
5. Quantities that are equal to the same quantity or to equal quantities are equal to each other.
6. A quantity may be substituted for its equal.
7. The whole is equal to the sum of all its parts.
8. The whole is greater than any of its parts.
9. One quantity is either greater than, equal to, or less than another quantity of the same kind.
10. If equals are added to or subtracted from unequals, or if unequals are multiplied or divided by equals, the results are unequal in the same order.
11. If unequals are added to unequals in the same order, the sums are unequal in the same order.
12. If unequals are subtracted from equals, the remainders are unequal in the reverse order.
13. If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.
14. If unequals are multiplied or divided by unequals, the results are unequal in the same order.
15. Like powers and like roots of equals are equal.

POSTULATES

1. Through two given points one and only one straight line can be drawn. (Or two points determine a line, or two segments coincide if their end points coincide.)
2. Two straight lines cannot intersect in more than one point.
3. A straight line is the shortest line that can be drawn between two points. (Or the shortest line that can be drawn between two points is the straight line joining them.)

4. A straight line may be extended indefinitely in both directions, or it may be limited by one point or by two points.
5. A geometric figure may be moved about in space freely without altering its size or shape.
6. All straight angles are equal.
7. All right angles are equal.
8. One and only one circle may be drawn with any given point as its center and any given line segment as its radius.
9. All radii of the same or of equal circles are equal.
10. A line segment can be bisected by one and only one point. (Or a line segment can have but one midpoint.)
11. An angle can be bisected by one and only one line.
12. Complements or supplements of equal angles are equal.
13. One and only one line can be drawn through a given point perpendicular to a given line.
14. The perpendicular is the shortest line that can be drawn from a point to a line.
15. One and only one line can be drawn through a given point parallel to a given line.

THEOREMS AND COROLLARIES

56. Geometric figures that can be made to coincide are congruent. Corresponding parts of congruent figures are equal.
57. An angle (symbol, \angle) is the figure formed by two lines that meet at a point. The point is called the *vertex* and the lines are called the *sides* of the angle.
58. Two angles that can be made to coincide are equal. Equal angles can be made to coincide.
59. A straight angle is an angle whose sides form one straight line extending in opposite directions from the vertex.
60. Adjacent angles are angles having a common vertex and a common side between them.
61. If one line intersects another line so as to form equal adjacent angles, these lines are perpendicular to each other (symbol, \perp) and the angles are right angles.
62. Two angles whose sum is a right angle are complementary angles. Each is the complement of the other.

63. Two angles whose sum is a straight angle are supplementary angles. Each is the supplement of the other.

64. When two lines intersect, either pair of non-adjacent angles is called *a pair of vertical angles* or *vertical angles*.

65. An acute angle is an angle that is less than a right angle. An obtuse angle is an angle that is greater than a right angle and less than a straight angle.

66. A polygon is a portion of a plane bounded by three or more line segments joined end to end. The segments are called the *sides* and the intersections of the segments the *vertices*. A *regular polygon* is both equilateral and equiangular.

67. A triangle (symbol, Δ) is a three-sided polygon.

68. An isosceles triangle is a triangle having two equal sides. A triangle having no two sides equal is a *scalene* triangle.

69. An obtuse triangle is one having an obtuse angle. An acute triangle has no obtuse angle or right angle.

70. An equilateral triangle is a triangle having three equal sides. (An equilateral triangle is, of course, isosceles.)

71. A quadrilateral is a four-sided polygon.

72. A square is a quadrilateral whose sides are equal and whose angles are equal.

79. A circle is a closed curve, all points of which are equidistant from a point within called the *center*.

80. Any portion of the curve is called an *arc*.

81. The distance from the center of the circle to the circle is called the *radius*. The term *radius* is also used to denote any one of the equal line segments (called *radii*) that can be drawn from the center to the circle. Note that by the definition of radius, all radii of the same circle are equal.

82. Equal circles are circles having equal radii. By the definition of equal circles, radii of equal circles are equal.

87. If two lines intersect, the vertical angles are equal.

89. Any two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other. (*s.a.s. = s.a.s.*)

90. Two triangles are congruent if a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other. (*a.s.a. = a.s.a.*)

94. In any isosceles triangle the angles opposite the equal sides are equal.

95. An equilateral triangle is equiangular.

97. Two triangles are congruent if the three sides of one are equal respectively to the three sides of the other. (*s.s.s. = s.s.s.*)

99. If two angles of a triangle are equal, the sides opposite the equal angles are equal.

100. An equiangular triangle is equilateral.

105. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other. (*h.a. = h.a.*)

107. Two right triangles are congruent if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other. (*h.s. = h.s.*)

110. Parallel lines are straight lines in the same plane which cannot meet however far produced.

111. Straight lines in the same plane perpendicular to the same line are parallel.

112. If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other.

113. Two lines parallel to a third line are parallel to each other.

114. A straight line that intersects two or more straight lines is called a transversal of those lines.

115. Angles that are on the same side of the transversal and on the same side of the lines cut by the transversal are *corresponding angles*.

Angles between the two lines and on opposite sides of a transversal are *alternate interior angles*.

Angles on opposite sides of a transversal and outside of the two lines are *alternate exterior angles*.

Angles between the lines and on the same side of the transversal are *consecutive interior angles*.

117. If two straight lines in the same plane are cut by a transversal making the alternate interior angles equal, the lines are parallel.

118. If two straight lines in the same plane are cut by a transversal making the corresponding angles equal, the lines are parallel.

119. If two straight lines in the same plane are cut by a transversal making the alternate exterior angles equal, the lines are parallel.

120. When two straight lines in the same plane are cut by a transversal, if the consecutive interior angles are supplementary, the lines are parallel; if not, the lines are not parallel.

122. If two parallel lines are cut by a transversal, the alternate interior angles are equal.

123. If two parallel lines are cut by a transversal, the corresponding angles are equal.

124. If two parallel lines are cut by a transversal, the alternate exterior angles are equal.

125. If two parallel lines are cut by a transversal, the consecutive interior angles are supplementary.

129. If the initial and terminal sides of one angle are parallel respectively to the initial and terminal sides of another, the angles are equal.

130. If the initial and terminal sides of one angle are parallel respectively to the terminal and initial sides of another, the angles are supplementary.

132. The sum of the angles of a triangle equals a straight angle.

133. An exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles and is greater than either of them.

134. If two angles of one triangle are equal to two angles of another, the third angles are also equal.

135. If the sum of two angles of a triangle equals the third angle, the third angle is a right angle.

136. The bisectors of any two angles of a triangle meet.

138. If two angles and a side of one triangle are equal respectively to two angles and the corresponding side of another, the triangles are congruent.

139. If the initial and terminal sides of one angle are perpendicular respectively to the initial and terminal sides of another, the angles are equal.

140. If two lines are perpendicular respectively to two non-parallel lines, they are not parallel.

143. A quadrilateral having both pairs of opposite sides parallel is called a *parallelogram*.

A parallelogram whose sides are equal is called a *rhombus*.

A parallelogram whose angles are right angles is called a *rectangle*.

A rectangle whose sides are equal is called a *square*.

A quadrilateral having one and only one pair of parallel sides is called a *trapezoid*. If the non-parallel sides of a trapezoid are equal, it is an *isosceles trapezoid*.

Any other quadrilateral is called an *irregular quadrilateral*.

- 145.** The opposite sides of a parallelogram are equal and the opposite angles are equal.
- 146.** The segments of parallel lines cut off by parallel lines are equal.
- 148.** Parallel lines are everywhere equidistant.
- 149.** The diagonals of a parallelogram bisect each other.
- 150.** If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.
- 151.** If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.
- 154.** If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.
- 155.** Any two consecutive angles of a parallelogram are supplementary.
- 157.** If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.
- 159.** If parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.
- 160.** If a line parallel to one side of a triangle bisects another side, it bisects the third side also.
- 161.** If a line joins the midpoints of two sides of a triangle, it is parallel to the third side and equal to half the third side.
- 162.** If a line parallel to the base of a trapezoid bisects one of the other sides, it bisects the opposite side and is equal to half the sum of the bases.
- 164.** The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.
- 167.** The sum of the interior angles of a polygon of n sides is $(n - 2)$ straight angles.
- 168.** If the angles of a quadrilateral are all equal, each is a right angle.
- 169.** Each angle of a polygon having equal angles is $\frac{(n - 2)}{n}$ straight angles.
- 171.** The sum of the exterior angles of a polygon made by producing each of its sides in succession is equal to two straight angles.
- 172.** If two sides of a triangle are unequal, the angles opposite those sides are unequal and the angle opposite the greater side is the greater.

174. If two angles of a triangle are unequal, the sides opposite these angles are unequal and the side opposite the greater angle is the greater.

176. If two sides of one triangle are equal respectively to two sides of another, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.

178. If two sides of one triangle are equal respectively to two sides of another, but the third side of the first triangle is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

182. (1) A straight line drawn through the center of a circle and terminating in its circumference is called its *diameter*. A diameter consists of two *radii*.

(2) A straight line having its end points on a circle is called a *chord*.

(3) An angle formed by two radii is called a *central angle*.

(4) The arc that is cut off by the sides of a central angle is called its *intercepted arc*.

184. Equal central angles of the same or equal circles have equal chords and equal arcs.

185. The greater of two central angles in a circle or in equal circles forms the greater chord, and conversely.

186. The greater of two central angles in a circle or in equal circles forms the greater arc, and conversely.

188. Equal chords in the same or equal circles have equal central angles and equal arcs.

189. The greater of two unequal chords in the same or in equal circles has the greater arc, and conversely.

191. Equal arcs of the same or equal circles have equal central angles and equal chords.

192. A *locus* is the path of a point moving so as constantly to fulfill a given condition (such as being 1 inch from a fixed point). All points that lie on the locus fulfill the given condition, and all points that fulfill the given condition lie on the locus.

197. The locus of points equidistant from two points is the perpendicular bisector of the line joining them.

200. The locus of points equidistant from two given intersecting lines is a pair of lines that bisect the angles formed by them.

- 204.** Through any three given points, not lying in a straight line, one circle and only one can be drawn.
- 209.** The perpendicular bisector of a chord passes through the center of the circle.
- 211.** A diameter perpendicular to a chord bisects the chord and the arcs of the chord.
- 213.** A diameter that bisects a chord that is not a diameter is perpendicular to it.
- 215.** The perpendicular bisectors of the sides of a triangle meet in a point.
- 217.** The bisectors of the angles of a triangle meet in a point.
- 220.** The altitudes of a triangle meet in a point.
- 221.** The point of intersection of the altitudes of a triangle is called the *orthocenter* of the triangle; the point of intersection of the perpendicular bisectors of the sides is called the *circumcenter*; the point of intersection of the bisectors of the angles is called the *incenter*.
- 222.** The intersection of the medians of a triangle is called the *centroid* of the triangle.
- 223.** The medians of a triangle meet in a point which is two thirds of the distance from each vertex to the midpoint of the opposite side.
- 225.** If a line is perpendicular to a radius at its outer end, the line is tangent to the circle.
- 226.** If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.
- 227.** If a line is perpendicular to a tangent at the point of contact, it passes through the center of the circle.
- 230.** The tangents to a circle from an external point are equal and make equal angles with the line joining the point to the center.
- 232.** If two parallel lines intersect a circle or are tangent to it, they intercept equal arcs.
- 239.** An inscribed angle is measured by half its intercepted arc.
- 240.** An angle inscribed in a semicircle is a right angle.
- 241.** The opposite angles of an inscribed quadrilateral are supplementary.
- 242.** Angles inscribed in the same arc, or in equal arcs, of a circle are equal.

243. An angle formed by a tangent and a chord drawn from the point of contact is measured by half its intercepted arc.
244. An angle formed by two chords intersecting within a circle is measured by one half the sum of an intercepted arc and that of its vertical angle.
246. An angle formed by two secants, by a secant and a tangent, or by two tangents, drawn to a circle from an external point is measured by half the difference between its intercepted arcs.
248. Equal chords in the same circle or in equal circles are equally distant from the center.
249. In the same or equal circles chords equally distant from the center are equal.
251. The greater of two unequal chords in the same or equal circles is nearer the center; and, conversely, the chord nearer the center is the greater.
252. A diameter of a circle is greater than any other chord.
264. The area of a rectangle is the product of its base and altitude.¹
268. The area of a parallelogram is the product of its base and altitude.
269. Parallelograms with equal bases and equal altitudes are equivalent.
271. The area of a triangle is half the product of its base and altitude.
272. Triangles with equal bases and equal altitudes are equivalent.
274. The area of a trapezoid is half the product of the altitude and the sum of the bases.
276. The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.
277. The square on either side of a right triangle is equal to the difference between the square on the hypotenuse and the square on the other side.
281. Transform a given polygon into an equivalent triangle.
285. A statement of the equality of two ratios is called a *proportion*. A proportion may be written in the form $a : b = c : d$ or, preferably, in the form $\frac{a}{b} = \frac{c}{d}$.

¹ The above reasoning assumes the base and the altitude of the rectangle to be commensurable. The case in which they are incommensurable is not given in this text.

In a proportion, the four parts (a , b , c , and d in the above proportion) are called the *terms*. The first and third terms (a and c) are called the *antecedents*; the second and fourth (b and d) are sometimes called the *consequents*.

The first and fourth terms (a and d) are called the *extremes* and the second and third (b and c) the *means*.

In the proportion $a : b = b : c$, the term b is called the *mean proportional* between a and c .

291. If a line is drawn through two sides of a triangle parallel to the third side, it divides the two sides proportionally.

292. If a line is drawn through two sides of a triangle parallel to a third side, either side is to one of its segments as the other side is to the corresponding segment.

293. Corresponding segments cut off on two transversals by a series of parallels are proportional.

298. If a line divides two sides of a triangle proportionally, it is parallel to the third side.

299. The bisector of an interior angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides.

302. If the bisector of an exterior angle of a triangle meets the opposite side produced, it divides that side externally into segments that are proportional to the adjacent sides.

304. Two polygons that are mutually equiangular and whose corresponding sides are proportional are said to be similar polygons.

307. Two mutually equiangular triangles are similar.

308. If two angles of one triangle are equal respectively to two angles of another, the triangles are similar.

309. If an acute angle of one right triangle is equal to an acute angle of another, the triangles are similar.

310. If two triangles have their sides respectively parallel to one another, the triangles are similar.

311. If two triangles have an angle of one equal to an angle of the other and the including sides proportional, the triangles are similar.

313. If two triangles have their sides respectively proportional, they are similar.

315. If in a right triangle the perpendicular is drawn from the vertex of the right angle of the hypotenuse,

I. The two triangles thus formed are similar to the given triangle and to each other.

II. The perpendicular is the mean proportional between the segments of the hypotenuse.

III. Each side of the given triangle is the mean proportional between the hypotenuse and the adjacent segment.

316. The perpendicular from any point on a circle to a diameter of the circle is the mean proportional between the segments of the diameter.

324. (I) The area of one rectangle is to the area of another as the product of the dimensions of the first is to the product of the dimensions of the second.

(II) Rectangles with equal bases are to each other as their altitudes.

(III) Rectangles with equal altitudes are to each other as their bases.

326. Two similar rectangles are to each other as the squares of their bases or as the squares of their altitudes.

328. Any two triangles or parallelograms are to each other as the products of their bases and altitudes.

329. Triangles or parallelograms with equal altitudes are to each other as their bases.

330. Triangles or parallelograms with equal bases are to each other as their altitudes.

332. If an angle of one triangle is equal to an angle of another, the triangles are to each other as the products of the sides forming the equal angles.

334. The areas of two similar triangles are to each other as the squares of any two corresponding sides.

336. If two polygons are similar, they can be separated into the same number of triangles, similar each to each and similarly placed.

337. If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.

339. The areas of two similar polygons are to each other as the squares of any two corresponding sides.

341. If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.

342. The product of the segments of any chord through a fixed point within a circle is constant.

344. If from a point outside a circle a secant and a tangent are drawn, the tangent is the mean proportional between the secant and its external segment.

345. The product of any secant from a fixed point outside a circle and its external segment is constant.

347. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

357. If a circle is divided into any number of equal arcs, the chords of these arcs form a regular inscribed polygon; and the tangents at the points of division form a regular circumscribed polygon.

359. A circle can be circumscribed about any regular polygon.

361. A circle may be inscribed in any regular polygon.

362. The center of the circle inscribed in or circumscribed about a polygon is called the *center* of the polygon.

363. The radius of the circle circumscribed about a polygon is called the *radius* of the polygon.

364. The radius of the circle inscribed in a regular polygon is called the *apothem* of the polygon.

365. An angle formed by adjacent radii of a polygon is called an *angle at the center* of the polygon.

366. Angles at the center of a regular polygon are equal, and each is supplementary to an interior angle of the polygon.

367. An equilateral polygon inscribed in a circle is regular.

369. Two regular polygons of the same number of sides are similar.

370. The areas of two regular polygons of the same number of sides are to each other as the squares of any two corresponding sides.

372. The perimeters of two regular polygons of the same number of sides are to each other as their radii and also as their apothems.

374. The area of a regular polygon is half the product of its apothem and its perimeter.

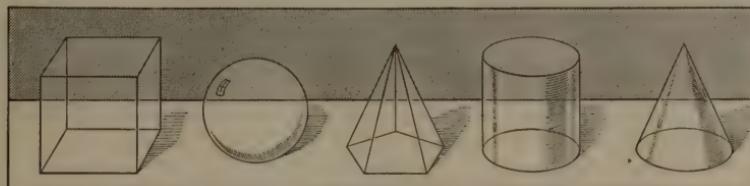
375. The areas of two regular polygons of the same number of sides are to each other as the squares of their radii and also as the squares of their apothems.

380. A pentadecagon is a polygon of fifteen sides.
389. The area of a circle is half the product of its radius and its circumference.
391. The ratio of the circumference of any circle to its radius is constant.
392. The ratio of the circumference of any circle to its diameter is constant.
394. Two circumferences have the same ratio as their radii.
395. In any circle $c = \pi d = 2 \pi r$.
396. The area of a circle is π times the square of the radius. $A = \frac{1}{2}rc = \frac{1}{2}r \times 2\pi r = ?$
397. The areas of two circles are to each other as the squares of their radii.
405. The area of a sector whose radius is r and whose arc contains n degrees is $\frac{n}{360} \times \pi r^2$.

MODERN SOLID GEOMETRY

BOOK SIX

LINES AND PLANES IN SPACE



407. Solid geometry. In plane geometry we have considered the relations of points and lines in a single plane. In solid geometry we are no longer limited in our discussion to a single plane, but will consider the relations between points, lines, planes, and curved surfaces in space. Book Six deals with lines and planes in space.

By a *plane in space* is meant the same sort of plane on which all the figures of plane geometry are considered to be drawn. A plane is something which has length and width but no thickness. It is like the surface of a pane of glass.

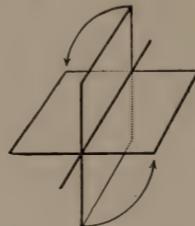
408. Definition. A *plane* (sometimes called a *plane surface*) is a surface such that a straight line joining any two points in the plane lies wholly in the plane.

Just as in plane geometry we discussed parallel lines which were drawn in a single plane, so in solid geometry we may discuss parallel planes in space, as for example the front and back surfaces of a book. And just as we may place a pane of glass in any position in space — horizontally, vertically, or inclined — so we may think of placing a plane in space.

409. Determining the position of a plane. Let us consider the ways in which the position of a plane may be determined.

If you were given the position of one point in space, as for example the corner of a table, and a thin pane of glass were brought into contact with this point, would it be rigid — that is, fixed?

Suppose you were given two points in space, as for example two corners of a table. Suppose a pane of glass were brought into contact with these two points. Would



its position be determined? That is, would it be rigid? Obviously not, for it might rotate upon the edge of the table as an axis.

Suppose, however, you were given three points in space, as for example the two corners of the table and the top of a pencil held upright on the table. Suppose the pane of glass were brought into contact with these three points. Would its position be determined? That is, would it be rigid? If so, then we may lay down this principle:

Three points not in a straight line determine (the position of) a plane.

Is it possible to “pass a plane” through any given line and any given external point? If so, would the plane be rigid? That is, suppose the line to be the edge of a table and the point to be the point of a pencil. Can a pane of

glass be brought into contact with the line and the point? Obviously so, for if we bring the pane of glass into contact with the edge of the table, it can be rotated until it is in contact with the point. We may therefore state the following variation of the preceding principle:

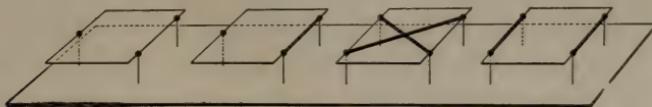
A line and a point not on the line determine a plane.

Do you think a plane could be passed through any two intersecting lines? If so, would the plane be rigid? Suppose we have two lines, AB and AC , intersecting at A , and a plane is passed through AB and rotated to contain the point C . Will it then contain the line AC ? (See the definition of a plane.) Would the plane be rigid?

Two intersecting lines determine a plane.

In plane geometry we define parallel lines as lines that lie in the same plane and that cannot meet, however far produced. Is a plane determined by two parallel lines? If a pane of glass were placed in contact with two parallel bars, would it be rigid? Obviously so, for its position would be fixed by one of the lines and any point of the other.

Two parallel lines determine a plane.



We may sum up as a postulate the principles we have found to apply to determining the position of a plane.

Postulate 17. (1) *Three points not in a straight line determine a plane.*

(2) *A line and a point not on the line determine a plane.*

(3) *Two intersecting lines determine a plane.*

(4) *Two parallel lines determine a plane.*

410. Intersecting planes. In plane geometry we have found that two lines intersect in a point. What is the nature of the intersection of two planes, such as the north wall and the east wall of a room? Can you give another illustration of the intersection of two planes? Suppose two planes, which let us call p and p' , each contain point A and point B . Will the line joining A and B lie wholly in plane p ? (See § 408, the definition of a plane.) Will it lie wholly in plane p' ? If so, it is their intersection, since no other point can lie in both planes. We may therefore state:

Postulate 18. *The intersection of two planes is a straight line; two planes can intersect in only one line.*

411. The postulates of plane geometry. Let us consider the postulates that were given on pages 32 and 33 of the *Plane Geometry* and see whether they apply to solid geometry. Postulate 1 reads, "Through two given points, one and only one straight line can be drawn." First of all, what would we mean by drawing a line in space, as for example from the corner of a table to a given spot on the floor? We may think of drawing a line through space as analogous to adjusting a very fine straight wire of indefinite length so as to be in contact with the two points. If a fine straight wire were adjusted to be in contact with the two points mentioned, would its position be fixed? If so, then Postulate 1 still holds good for solid geometry.

Let us consider Postulate 2. If two fine straight wires are placed one across the other anywhere in space, can they come in contact in more than one point?

May we consider Postulate 2 as true also for solid geometry? Does Postulate 3 hold for solid geometry? Postulate 4? Do Postulates 5 to 12 hold for solid geometry?

412. Perpendicular lines. Is Postulate 13 true for solid geometry? It reads, "One and only one line can be drawn through a given point perpendicular to a given line." Let us consider one of the edges of a square post as a line in space, and let us take a given point not on the line. Can a fine straight wire be adjusted so as to be in contact with the point and perpendicular to the line? Is there more than one such position? If not, Postulate 13 holds when the given point is not on the line.

Let us now consider the given point as being on the line. Is it possible to adjust a fine straight wire so as to be perpendicular to the edge of the post at the given point? Is there more than one such position? If so, then Postulate 13 as stated does not hold for solid geometry when the given point is on the given line and it must be restated for solid geometry.

Postulate 13. (Restated.) *One and only one line can be drawn in a given plane through a given point in the plane perpendicular to a given line in the plane.*

This, of course, is just what was meant in the statement of Postulate 13 for plane geometry.

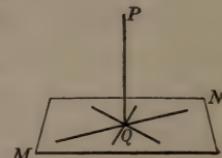
All the postulates of plane geometry hold good for solid geometry except as noted above with regard to Postulate 13.

413. Line intersecting plane. We have said in *Plane Geometry* that one straight line can intersect another at only one point. Can a straight line intersect a plane in more than one point? Consider, for example, a fine straight wire passing through the level surface of a pan of water.

Postulate 19. *A straight line can intersect a plane in only one point.* (If a line and a plane have two points in common, where must the line lie?)

414. Perpendicular line and plane. In plane geometry we have dealt with lines perpendicular to other lines. Let us consider what we should mean by saying that a line is perpendicular to a plane. Let us assume that we are holding a flat pane of glass in a tilted position. What would it mean to say that we were holding a fine straight wire perpendicular to the surface of the pane of glass? Let us agree upon the following definition:

415. Definition. A line and plane are said to be perpendicular to each other if the line is perpendicular to every line in the plane through its foot. Thus PQ is perpendicular to plane MN , and MN is perpendicular to PQ , if PQ is perpendicular to every line in MN that passes through Q .



Since the above statement is a definition, and since definitions are reversible, it follows also that if a line is perpendicular to a plane, it is perpendicular to every line in the plane through its foot; or, in other words, if a plane is perpendicular to a given line, all the lines in the plane drawn through the intersection of the line and plane are perpendicular to the given line.

A line that intersects a plane, but not perpendicularly, is said to be *oblique* to the plane.

Ex. 1. Can you pass more than one plane through a line?

Ex. 2. Can you pass a plane through any two lines?

Ex. 3. Is a plane determined by any three points?

Ex. 4. Find in your schoolroom several cases of intersection of two planes.

Ex. 5. Can you pass a plane through any two intersecting lines?

Ex. 6. When do three points determine a plane?

Ex. 7. Can you determine whether or not a surface is a plane by one application of a straightedge?

Ex. 8. Can a line have two points common to itself and a plane?

Ex. 9. Can a line intersect a plane in more than one point?

Ex. 10. Is a triangle a plane figure? That is, must it lie in a plane?

Ex. 11. Is any four-sided figure necessarily a plane figure?

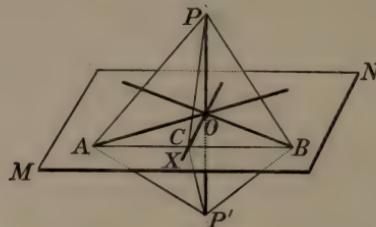
416. Drawing a line in space through a given point perpendicular to a given line. Suppose we have a given line in space and a given point not on the line. How may we draw a line through the point perpendicular to the given line? Obviously, if we pass a plane through the given line and the given point, we may then draw the required perpendicular in the plane in the usual way. Therefore, in solid geometry we may assume that it is entirely possible to draw a line in space from a given point perpendicular to a given line.

Obviously, if the point lies on the line, we may pass any plane through the line and in the plane erect a perpendicular to the line at the given point.

417. Passing a plane perpendicular to a line. Suppose we have a given line in space and a given point on the line and wish to pass a plane perpendicular to a given line at the given point. How may this be done? By the definition of perpendicular line and plane (§ 415) a plane is perpendicular to a line if it contains all the perpendiculars to the line at the intersection of the line and plane. Suppose a plane is placed so as to contain two such perpendiculars to the given lines. Would it contain all the others? Proposition 1 gives the answer to this question.

Proposition 1. Lines in a Plane Perpendicular to a Line

418. Theorem. *If two given lines are each perpendicular to a third, any line in the plane of the two given lines through their intersection is perpendicular to the third.*



Given AO and BO each \perp to PO at O , and MN the plane of AO and BO . Given also XO , any other line through O lying in the plane MN .

To prove that XO is \perp to PO .

The plan is to extend PO to P' , making $P'O =$ to PO , and show that a point C on XO is equidistant from P and P' .

Proof. Draw AB intersecting XO at C . Extend PO to P' , making $P'O =$ to PO . Draw PA , PC , PB , $P'A$, $P'C$, and $P'B$. In $\triangle PAB$ and $\triangle P'AB$,

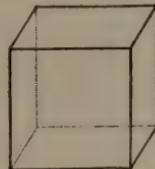
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| <ol style="list-style-type: none"> 1. $PA = P'A$, and $PB = P'B$. 2. Hence $\triangle PAB \cong \triangle P'AB$, 3. and $\angle PAB = \angle P'AB$. 4. Hence $\triangle PAC \cong \triangle P'AC$, 5. and $PC = P'C$. 6. $\therefore CO$ is \perp to PP'; that is,
PO is \perp to OC. | <ol style="list-style-type: none"> 1. § 197. 2. s.s.s. = s.s.s. 3. c.p.c.t.e. 4. s.a.s. = s.a.s. 5. c.p.c.t.e. 6. § 197. |
|---|--|

Therefore, if two given lines are each perpendicular to a third, any line in the plane of the two given lines through their intersection is perpendicular to the third.

419. Corollary. *If a line is perpendicular to two intersecting lines at their point of intersection, it is perpendicular to the plane of the lines.¹*

420. Figures in solid geometry. In drawing the figures of solid geometry, which must represent three dimensions, a difficulty arises in representing the lines which recede from the eye. The following method of drawing figures for solid geometry has been found practical :

Let us assume that we wish to draw a cube on the blackboard. We will first place it so that the front face of the cube is parallel with the blackboard. This we will represent as a square. The edges of the cube which recede from the eye we will represent by slanting lines, as shown in the figure. These should make an angle of about 60° with the horizontal.



The simple rules for drawing figures in solid geometry are :

1. Figures that are parallel to the blackboard should be drawn exactly as they appear (horizontal lines, horizontal ; vertical lines, vertical).
2. Horizontal lines which recede from the eye should be indicated by slanting lines making an angle of about 60° with the horizontal.

You will note that the directions given above make no provision for what is known as perspective ; namely, the tendency for parallel lines receding from the eye to appear to converge at a point on the horizon. The figures shown in this book are drawn with a slight perspective, since this aids in the interpretation of the figures. However, for practical purposes the student may disregard perspective and draw parallel receding lines at the same angle.

¹ It will be assumed hereafter that the student will prove all corollaries.

421. Passing a plane perpendicular to a given line at a given point on the line. Let us return now to the problem of placing a plane in space so that it will be perpendicular to a given line at a given point on the line. By § 418 it follows that if we can adjust a plane so as to contain any two lines perpendicular to the given line at the given point, the plane will contain all the perpendiculars to the line at the point and hence be perpendicular to the line at that point. We have considered in § 416 the means by which these two perpendicular lines might be drawn, and since any two intersecting lines determine the position of a plane, we could easily adjust the plane so as to contain these two perpendicular lines. Corollary 422 follows from the above considerations.

422. Corollary. *A plane containing two lines, each perpendicular to a given line at a point, is perpendicular to the given line at that point.*

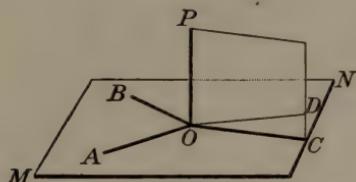
This corollary is logically the same as § 419. What is the hypothesis in § 419? What is the hypothesis in § 422? What is the conclusion in § 419? What is the conclusion in § 422?

Note that to say that a plane is perpendicular to a line is the same as saying that the line is perpendicular to the plane. This second form of the corollary is stated, since it is somewhat more convenient for use in subsequent proofs.

423. Converse of Proposition 1. Let us consider the converse of Proposition 1. If a line is perpendicular to a plane, do all the lines that are perpendicular to the given line at its intersection with the plane lie in the plane? Proposition 2 answers this question.

Proposition 2. Lines Perpendicular to a Line

424. Theorem. *All the perpendiculars to a given line at a given point lie in a plane perpendicular to the line at this point.*



Given AO , BO , CO , etc., each \perp to PO at point O .

To prove that AO , BO , CO , etc., lie in a plane \perp to PO at O .

The plan is to show that, if we pass a plane through any two of the lines, any third line will be found to lie in this plane.

Proof.

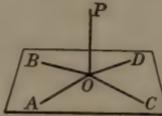
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|--|-------------------------|
| 1. Any two of the lines, as AO and BO , determine a plane. (Call the plane MN .) | 1. Post. 17. <i>Two</i> |
| 2. MN is \perp to PO at O . | 2. § 422. |
| 3. Pass a plane through PO and CO . | 3. Post. 17. |
| 4. It will intersect MN in some line.
(Since we have not proved that CO is the intersection, let us call the intersection DO .) | 4. Post. 18. |
| 5. Hence PO is \perp to DO . | 5. § 415. |
| 6. But CO is \perp to PO and in the same plane as DO and PO . | 6. By hyp. and constr. |
| 7. Hence CO coincides with DO . | 7. Post. 13. |
| 8. $\therefore CO$ lies in plane MN . | 8. Ax. 6. |

Likewise any other perpendicular to PO at O can be shown to lie in MN .

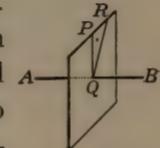
Therefore, all the perpendiculars . . .

425. Corollary. *One and only one plane may be passed perpendicular to a given line at a given point on the line.*

Let O be the given point in the given line PO . Draw OA and $OB \perp$ to PO at O . They determine one plane \perp to PO at O . (§ 422.) Assume for the moment that some other plane is \perp to PO at O . Draw any two lines OC and OD in this plane. OC and OD are both \perp to PO . (§ 415.) But all the perpendiculars to PO at O lie in one plane and so the second plane must coincide with the first. (§ 424.)



426. Passing a plane perpendicular to a line through a given external point. Suppose we have a given line AB in space and a given point P not on the line. Can you think how to pass a plane through P perpendicular to AB ? Obviously it will contain the line PQ from P perpendicular to AB . Why? Will it contain any other perpendicular to AB at Q , as QR ? If so, then we can easily pass the required plane by dropping a perpendicular from P to AB , erecting QR another perpendicular to AB at Q , and pass a plane through PQ and QR .



Ex. 1. If each of three lines is perpendicular to both of the other two, must they all pass through a common point?

Ex. 2. If each of three lines is perpendicular to the other two, can a fourth line be drawn perpendicular to the first three?

Ex. 3. AO and BO are each perpendicular to PO at O . Plane MN contains AO and BO . Is MN perpendicular to PO ?

Ex. 4. PO is oblique to plane MN . Can it be perpendicular to AO and BO , lines in plane MN ?

Ex. 5. CO lies in the plane of AO and BO . If PO is perpendicular to CO , is it perpendicular to the plane of the given lines?

427. Corollary. *Through a given point not on a given line one and only one plane can be passed perpendicular to the line.*

Let AB be the given line and P be the given point. Draw $PQ \perp$ to AB and from Q draw QR , some other perpendicular to AB . Then PQ and QR determine a plane \perp to AB at Q . (\S 422.) Hence one perpendicular plane can be drawn to AB from P .

Let us assume for the moment that another plane can be drawn through $P \perp$ to AB . Let this plane intersect AB at Q' . Draw PQ' and RQ' . Since PQ and PQ' are both \perp to AB , they coincide (Post. 13). Hence Q' coincides with Q , and PQ' and RQ' coincide with PQ and RQ , respectively (Post. 1). Hence only one plane can be passed through $P \perp$ to AB (Post. 17).

Ex. 1. Find in your schoolroom several cases of the intersection of three planes.

Ex. 2. Do three planes intersect in a line? 

Ex. 3. In space can there be more than one line \perp to a given line at a given point? How many?

Ex. 4. Can you make a general statement covering both \S 425 and \S 427?

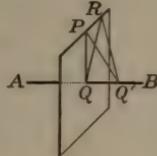
Ex. 5. Complete: If a line is to two lines at their point of , it is perpendicular to the of these lines.

Ex. 6. Complete: If a line is perpendicular to a plane, the plane is to the at their point of .

Ex. 7. Complete: If a line is perpendicular to another line, it lies in a to the at the point of .

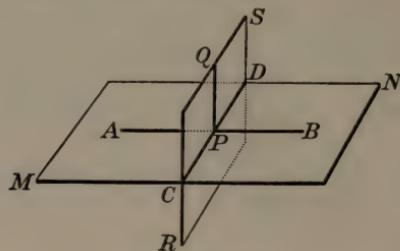
Ex. 8. P is a point without AB . PQ is perpendicular to AB at Q . QR is another line perpendicular to AB at Q . Complete: PQ and QR a plane to AB at .

Ex. 9. How could you use \S 422 to determine whether or not a rod is held perpendicularly to the floor?



Proposition 3. Line Perpendicular to a Plane

428. Problem. Draw a line through a given point in a plane perpendicular to the plane.



Given point P in plane MN .

Required to draw a line through $P \perp$ to plane MN .

The plan is to draw a line that is perpendicular to each of two lines through P in plane MN .

Steps in construction. (1) Draw a line AB through P in plane MN . (2) Pass a plane RS through $P \perp$ to AB , intersecting MN in CD . (3) Draw PQ in $RS \perp$ to CD . PQ is the required perpendicular.

Proof.

- | | |
|---|---|
| 1. PQ is \perp to CD .
2. But PQ is \perp to AB .
3. $\therefore PQ$ is \perp to MN , the | 1. By construction.
2. § 415.
3. § 419. |
|---|---|
- plane of AB and CD .

Ex. 1. The plane of the blackboard in your room is vertical. Would a line perpendicular to it be horizontal? Would a plane?

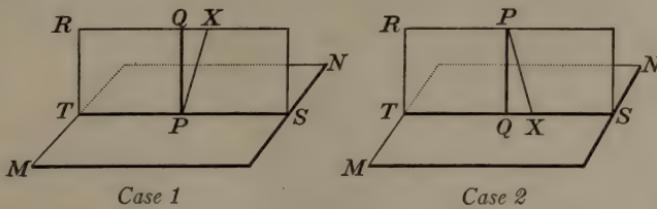
Ex. 2. Would all the lines perpendicular to a north-and-south line lie in an east-and-west direction?

Ex. 3. Would all the lines perpendicular to a vertical line be horizontal? all the planes?

Ex. 4. In the figure of § 428, is AB perpendicular to CD ?

Proposition 4. Line Perpendicular to a Plane

429. Theorem. Through a given point (internal or external to a plane) only one line can be drawn perpendicular to the plane.



Given plane MN and line $PQ \perp$ to MN . (Case 1 with P on MN , and Case 2 with P not on MN .)

To prove that no other line can be drawn through $P \perp$ to MN .

The plan is to show that any other line PX that is \perp to MN must coincide with PQ .

Proof. (The proof applies to both cases.)

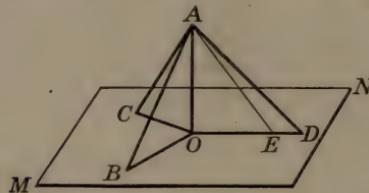
Assume for the moment that some other line through P , as PX , can be drawn \perp to MN .

- | | |
|--|---------|
| 1. PQ and PX determine a plane. (Call the plane RS .) | 1. Why? |
| 2. RS will intersect MN in a line. (Call the line ST .) | 2. Why? |
| 3. PQ and PX are both \perp to ST . | 3. Why? |
| 4. But this is impossible; therefore no other line, as PX , can be drawn \perp to MN . | 4. Why? |

Therefore, through a given point

Proposition 5. Lines Oblique to a Plane

430. Theorem. *Oblique lines drawn from a point in a perpendicular to a plane meeting the plane at equal distances from the foot of the perpendicular are equal; and, of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.*



Given $AO \perp$ to plane MN at O , and AB and AC any two oblique lines drawn from A to plane MN , making $BO = CO$, and AD drawn from A to MN , making $DO > CO$.

PART 1. To prove that $AB = AC$.

The plan is to show that AB and AC are c.p.c.t.

Proof. In right $\triangle AOB$ and AOC ,

- | | |
|---|---|
| 1. $AO = AO$,
2. and $OB = OC$.
3. Hence rt. $\triangle AOB \cong$ rt. $\triangle OAC$.
4. $\therefore AB = AC$. | 1. Why?
2. Why?
3. s.a.s. = s.a.s.
4. c.p.c.t.e. |
|---|---|

PART 2. To prove that $AD > AC$.

The plan is to show that $AD > AE$ which equals AC .

Proof. On OD lay off $OE =$ to OC ; draw AE .

- | | |
|--|--|
| 5. $AE = AC$.
6. AE lies in plane AOD .
7. $\angle AED > \angle ADE$.
8. Hence $AD > AE$.
9. $\therefore AD > AC$. | 5. Part 1.
6. Two points common.
7. Why?
8. § 174.
9. Ax. 6. |
|--|--|

Ex. 1. State the converse of Proposition 5. Is it true?

431. Corollary. *Equal oblique lines drawn from a point in a perpendicular to a plane meet the plane at points equidistant from the foot of the perpendicular.*

432. Corollary. *The perpendicular is the shortest line from a point to a plane.*

Ex. 1. Find two lines in your room that cannot meet and yet are not parallel.

Ex. 2. Has a plane three dimensions?

Ex. 3. How many dimensions has a point? a line? a plane? a solid?

Ex. 4. Can a line be perpendicular to more than two lines in a plane?

Ex. 5. If three lines are each perpendicular to a line in a plane, are they mutually parallel? Why?

Ex. 6. If a circle is drawn in a plane and a perpendicular to this plane is drawn to the center of the circle, is a point in this perpendicular the same distance from every point in the circle?

Ex. 7. If a line is perpendicular to a line of a plane, is it perpendicular to the plane?

Ex. 8. Can a carpenter determine by one application of his square that a piece of studding is perpendicular to the floor? by two applications?

Ex. 9. Can you pass two planes through a given point each perpendicular to a line?

Ex. 10. If two lines are each perpendicular to the same line, are they parallel?

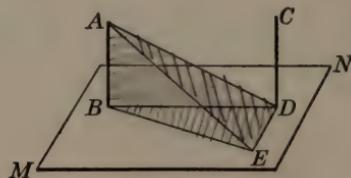
Ex. 11. If you fold down a corner of your book, is the crease straight?

NOTE. The student will prove his answers to all questions.

2/10/19

Proposition 6. Lines Perpendicular to a Plane Are Parallel

433. Theorem. Two lines perpendicular to the same plane are parallel.



Given AB and CD each \perp to plane MN as shown.

To prove that AB is \parallel to CD .

The plan is to prove that AB and CD lie in the same plane and are each \perp to BD .

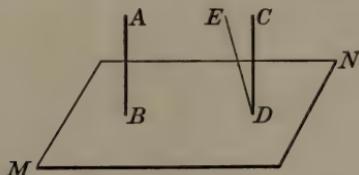
Proof. Draw BD , and through D draw DE in plane MN \perp to BD . Take A on AB and E on DE so that $DE = AB$. Draw AD , AE , and BE . In rt. $\triangle ABD$ and BDE ,

- | | |
|---|---|
| <ol style="list-style-type: none"> 1. $AB = DE$, and $BD = BD$. 2. Hence rt. $\triangle ABD \cong$ rt. $\triangle BDE$, 3. and $AD = BE$. <p style="text-align: center;">In $\triangle ADE$ and ABE.</p> <ol style="list-style-type: none"> 4. $AB = DE$ and $AE = AE$. 5. Hence $\triangle ADE \cong \triangle ABE$, 6. and $\angle ADE = \angle ABE$. 7. But $\angle ABE$ is a rt. \angle. 8. Hence $\angle ADE$ is a rt. \angle. 9. Hence ED is \perp to AD. 10. Hence AD, BD, and CD all lie in the same plane. 11. Since AB has two points common to the plane of AD, and BD, it lies in the plane. 12. $\therefore AB$ is \parallel to CD. | <ol style="list-style-type: none"> 1. Why? 2. s.a.s. = s.a.s. 3. c.p.c.t.e. <ol style="list-style-type: none"> 4. Why? 5. Why? * 6. c.p.c.t.e. 7. Why? 8. Why? 9. Why? 10. § 424. 11. Why? 12. § 111. |
|---|---|

Therefore, two lines perpendicular

Proposition 7. Converse of Proposition 6

434. Theorem. *If one of two parallel lines is perpendicular to a plane, the other also is perpendicular to the plane.*



Given AB and CD two \parallel lines of which AB is \perp to MN .

To prove that CD is \perp to MN .

Proof. Assume for the moment that CD is not \perp to MN .

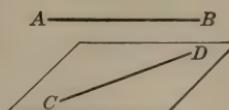
- | | |
|--|--|
| 1. Then some other line can be erected
\perp to MN at D . (Call this line DE .)
2. Then DE would be \parallel to AB .
3. $\therefore DE$ coincides with CD , which is \parallel to
AB by hypothesis. | 1. § 428.
2. § 433.
3. Post. 15. |
|--|--|

Therefore, if one of two parallel lines

435. Corollary. *If each of two lines is parallel to a third line, they are parallel to each other.*

They would each be perpendicular to the same plane by § 434 and parallel to each other by § 433.

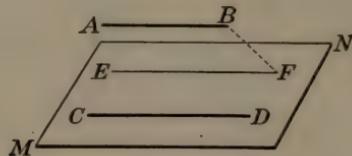
436. Parallel lines and planes. If a line cannot meet a plane however far they are produced, the line is parallel to the plane and the plane is parallel to the line. Likewise, if two planes cannot meet however far they are produced the planes are parallel to each other.



437. Skew lines. Two lines through which no one plane can be passed are called *skew lines*. AB and CD are skew lines.

Proposition 8. Line Parallel to a Plane

438. Theorem. *If two lines are parallel, every plane containing one of the lines and only one is parallel to the other.*



Given $AB \parallel CD$; and MN containing CD but not AB .
To prove that AB is \parallel to MN .

The plan is to show that the supposition that AB is not \parallel to plane MN leads to an impossibility. (Can you show this?)

Proof. Assume for the moment that AB is not \parallel to plane MN .

1. Then AB extended will meet plane MN at some point. (Call the point F .) Assume a line EF drawn through F in plane $MN \parallel$ to CD . | 1. § 436.
2. We now have two lines through $F \parallel$ to CD , which is impossible. Therefore AB cannot meet plane MN . | 2. Post. 15.

Therefore, if two lines are parallel

439. Corollary. *If a line is parallel to a plane, it is parallel to the intersection of that plane with any plane through the line.*

Ex. 1. If each of two planes passes through a line that is parallel to a third plane, are their intersections with the third plane parallel to each other?

Ex. 2. How do cement workers make the surface of the sidewalk a plane surface?

Proposition 9. Skew Lines and Parallel Plane

440. Theorem. *Through either of two skew lines one and only one plane can be passed parallel to the other line.*

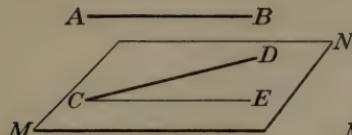


FIG. 1

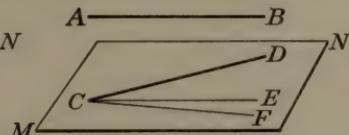


FIG. 2

Given two skew lines AB and CD .

To prove (1) that a plane can be drawn through $CD \parallel$ to AB and (2) that such a plane is the only plane through $CD \parallel$ to AB .

Proof of 1. Through C draw $CE \parallel$ to AB . (See Figure 1.)

- | | |
|--|---------------------------|
| 1. Pass a plane MN through CD and CE .
2. MN is \parallel to AB . | 1. Post. 17.
2. § 438. |
|--|---------------------------|

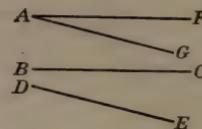
Proof of 2. (See Figure 2.) Assume for the moment that some plane other than MN (call it $M'N'$) can be passed through $CD \parallel$ to AB . Pass a plane PQ through AB and C , letting it intersect plane $M'N'$ in CF .

- | | |
|--|---|
| 1. CF does not coincide with CE , since the two planes intersect in CD .
2. Now CF is \parallel to AB .
3. But CE is \parallel to AB .
4. Therefore we have two lines through C in the plane of AB and $C \parallel$ to AB , which is impossible. Hence only one plane can be passed through $CD \parallel$ to AB . | 1. Post. 18.
2. § 439.
3. Construction.
4. Post. 15. |
|--|---|

Therefore, through either of two skew lines

441. Corollary. *Through a given point in space one plane and only one can be passed parallel to each of two skew lines, or else parallel to one line and containing the other.*

Through A , the given point, draw lines AF and $AG \parallel$ respectively to BC and DE , the given skew lines. Now BC is either \parallel to the plane of AF and AG or else is embraced by it. (\S 438.) Likewise DE is either \parallel to the plane of AF and AG or is embraced by it. Since BC and DE are skew lines, they cannot both be embraced by the plane of AF and AG . (\S 437.)



Ex. 1. Can two skew lines be perpendicular to each other?

Ex. 2. Can two skew lines be such that one is vertical and the other is horizontal? Can they both be vertical? both horizontal?

Ex. 3. Can two skew lines intersect?

Ex. 4. If two lines are not skew, are they parallel?

Ex. 5. Can two skew lines have the same direction?

Ex. 6. If a quadrilateral is skew (i.e., if it lies in two intersecting planes), do the lines joining the midpoints of the sides in order form a parallelogram?

Ex. 7. Do the lines joining the midpoints of the opposite sides of a skew quadrilateral intersect and bisect each other?

Ex. 8. Do the diagonals of a skew quadrilateral intersect?

Ex. 9. Do the perpendiculars to two intersecting planes from a point not in either of them lie in the same plane?

Ex. 10. Can a line always be drawn through a point not in either of two skew lines such that it cuts both of them?

Ex. 11. Do the perpendiculars to two skew lines from a point not on either of them lie in the same plane?

Ex. 12. If two lines from the same point to a plane have equal projections upon the plane, are they equal? Do they make equal angles with the plane?

Proposition 10. Parallel Planes

442. Theorem. *Two planes perpendicular to the same line are parallel, and conversely, if one of two parallel planes is perpendicular to a line, the other also is perpendicular to the line.*

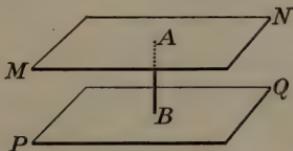


FIG. 1

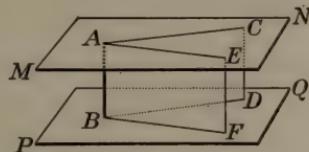


FIG. 2

PART 1. Given planes MN and PQ each \perp to AB .
(Fig. 1.)

To prove that MN is \parallel to PQ .

Proof. 1. If MN and PQ are not parallel, they will meet.

2. They cannot meet since, if they did, there would be two planes \perp to the same line through the same point (the meeting point).

3. $\therefore MN$ is \parallel to PQ .

PART 2. Given plane $MN \parallel$ to plane PQ and \perp to AB .
(Fig. 2.)

To prove that PQ is \perp to AB .

Proof. Pass a plane through AB intersecting MN and PQ in AC and BD respectively.

1. AC is \parallel to BD .

2. But AC is \perp to AB .

3. Hence BD is \perp to AB .

Now pass another plane through AB , intersecting MN and PQ in AE and BF respectively.

4. Similarly, BF is \perp to AB .

5. \therefore Plane PQ is \perp to AB .

1. § 436.

2. § 427.

3. Why?

1. Why?

2. § 415.

3. § 112.

4. 1, 2, 3.

5. § 422.

Therefore, two planes perpendicular

443. Corollary. *Through a given point not in a plane, one and only one plane can be passed parallel to the given plane.*

In the figure for Part 1 (p. 345) let PQ be the given plane and A the given point. Let AB be \perp to PQ . Since only one plane through A can be drawn \perp to AB , only one plane through A can be \parallel to PQ . (§ 425.)

EXERCISES

Ex. 1. Prove that the line joining two points equidistant from a plane and on the same side of it is parallel to the plane.

HINT. Is the figure formed a parallelogram?

Ex. 2. Through a given point in space one line and only one line may be drawn parallel to a given line.

HINT. Do the given point and the given line determine a plane? more than one? Can a line in this plane be drawn parallel to the given line? Can more than one?

Ex. 3. Theorem 442 may also be stated thus: If a ~~line~~ is \perp to each of two planes, the ~~lines~~ are \parallel , and conversely, if a line is \perp to one of two parallel planes, ~~the other~~.

Ex. 4. If a line intersects one of two parallel planes, must it intersect the other?

Ex. 5. Are all the other possible lines of Exercise 2 skew to the given line?

Ex. 6. If two planes are each parallel to a third plane, are they parallel to each other?

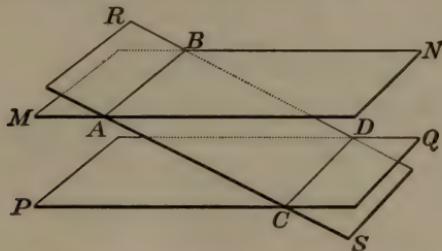
Ex. 7. The foot of the perpendicular from a point to a plane is called the projection of the point on the plane. Illustrate.

Ex. 8. The projection of a line segment on a plane is the line on the plane joining the projections of the ends of the given line segment. Illustrate.

Ex. 9. AB joins any point A in plane MN with any point B in plane PQ . MN is \parallel to PQ . Is the projection of AB on MN parallel to its projection on PQ ? equal?

Proposition 11. Intersection of Parallel Planes

444. Theorem. *If two parallel planes are cut by a third plane, the lines of intersection are parallel.*



Given MN and PQ , two \parallel planes each cut by plane RS in AB and CD respectively.

To prove that AB is \parallel to CD .

The plan is to prove that AB and CD which are in the same plane RS cannot meet, since MN is \parallel to PQ . (Can you prove it without referring to the following proof?)

Proof.

- | | |
|--|-------------------|
| 1. Since MN and PQ are parallel planes, line AB in MN and line CD in PQ cannot meet, | 1. § 436. |
| 2. and AB and CD lie in plane RS . | 2. By hypothesis. |
| 3. $\therefore AB$ is \parallel to CD . | 3. § 110. |

Therefore, if two parallel planes are cut by a third

Ex. 1. State the converse of Proposition 11. Is it true?

HINT. Can MN be rotated about AB ? Would it be parallel in more than one position?

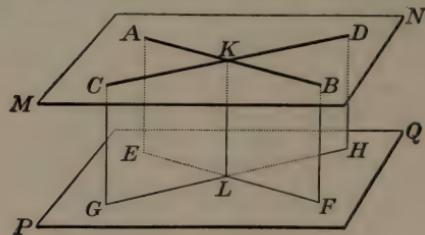
Ex. 2. In the figure of § 444, is $ABCD$ a parallelogram?

Ex. 3. Does a shed roof meet the tops of each pair of opposite sides in parallel lines?

Ex. 4. MN and PQ are two planes cut by the plane RS in AB and CD . Is AB parallel to CD ?

Proposition 12. Parallel Planes

445. Theorem. *If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to that plane.*



Given AB and CD , two lines intersecting at K , each \parallel to plane PQ ; and MN , the plane of AB and CD .

To prove that MN is \parallel to PQ .

The plan is to draw $KL \perp$ to PQ and prove that plane MN is \perp to KL and is therefore \parallel to plane PQ .

Proof. Draw $KL \perp$ to PQ , meeting PQ at L , and pass planes through AB and KL and through CD and KL intersecting PQ in EF and GH , respectively.

- | | |
|--|---|
| 1. EF is \parallel to AB , and GH is \parallel to CD .
2. But EF and GH are each \perp to KL .
3. Hence AB and CD are \perp to KL .
4. Hence MN is \perp to KL .
5. $\therefore MN$ is \parallel to PQ . | 1. § 444. ✓
2. § 415.
3. § 112.
4. § 422.
5. § 442. |
|--|---|

Therefore, if two intersecting lines are each parallel

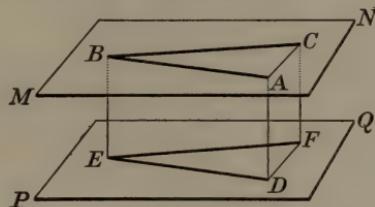
Ex. 1. Do all lines through a given point, that are parallel to a given plane, lie in a plane?

Ex. 2. Diagonals AC and BD of a parallelogram are \parallel to a plane MN . Are the sides of the parallelogram \parallel to MN ?

Ex. 3. Do two intersecting lines each parallel to a given plane determine a plane parallel to the given plane?

Proposition 13. Angles with Parallel Sides

446. Theorem. *If two angles not in the same plane have their initial and terminal sides respectively parallel, they are equal and their planes are parallel.*



Given $\angle ABC$ and $\angle DEF$, with initial side $AB \parallel$ to initial side DE and terminal side $BC \parallel$ to terminal side EF , and MN and PQ , the planes of $\triangle ABC$ and $\triangle DEF$, respectively.

To prove (1) that $\angle ABC = \angle DEF$ and (2) that plane MN is \parallel to plane PQ .

The plan is to lay off $BA =$ to ED and $BC =$ to EF , draw CA and FD , and prove that $\triangle ABC \cong \triangle DEF$.

Proof of 1. Lay off $BA =$ to ED and $BC =$ to EF and draw BE , CF , AD , AC , and DF .

- | | |
|--|---|
| 1. $ABED$ and $CBEF$ are parallelograms.
2. Hence CF and AD are $=$ and \parallel ,
3. and $ACFD$ is a parallelogram.
4. Hence $AC = DF$,
5. and $\triangle ABC \cong \triangle DEF$.
6. $\therefore \angle ABC = \angle DEF$. | 1. § 157.
2. Ax. 5 and § 435.
3. § 157.
4. Why?
5. s.s.s. = s.s.s.
6. c.p.c.t.e. |
|--|---|

Proof of 2.

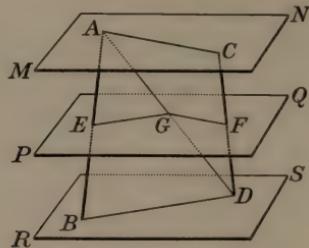
- | | |
|--|----------------------|
| 7. If MN is not \parallel to PQ , it will intersect
PQ in a line \parallel to both ED and EF .
8. But this is impossible.
Therefore MN is \parallel to PQ . | 7. § 439.
8. Why? |
|--|----------------------|

Therefore, if two angles



Proposition 14. Parallel Planes Intercepting Lines

447. Theorem. *If two lines are cut by three parallel planes, the corresponding segments are proportional.*



Given AB and CD , two lines cut by \parallel planes MN , PQ , and RS at A , E , B , and C , F , D , respectively.

To prove that $\frac{AE}{EB} = \frac{CF}{FD}$.

The plan is to form two triangles ABD and ADC with a common side AD , each with a line \parallel to the base, and form two proportions of which one ratio will be common. (Can you write out the proof without referring to what follows?)

Proof. Draw AD intersecting PQ at G . Through AB and AD pass a plane intersecting PQ in EG and RS in BD . Pass a plane through AD and CD intersecting PQ in GF and MN in AC .

- | | |
|---|---|
| 1. EG is \parallel to BD .
2. And FG is \parallel to AC .
3. Hence $\frac{AE}{EB} = \frac{AG}{GD}$.
4. And $\frac{CF}{FD} = \frac{AG}{GD}$.
5. $\therefore \frac{AE}{EB} = \frac{CF}{FD}$. | 1. § 444.
2. Why?
3. Why?
4. Why?
5. Ax. 5. |
|---|---|

Therefore, if two lines are cut

448. Dihedral angles. The figure formed by two intersecting lines is called a *plane angle*. The figure formed by two planes that intersect is called a *dihedral angle*. The planes are called the *faces* of the dihedral angle, and the intersection of the planes is called its *edge*. When two lines intersect, they form two pairs of equal vertical angles; similarly, when two planes intersect, they form two pairs of equal *vertical dihedral angles*.

The size of a dihedral angle may also be defined as the amount of turning of a plane about a line in the plane from one position to another.

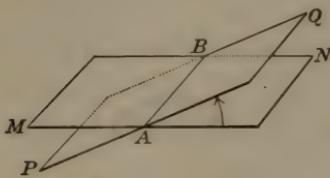


FIG. 1

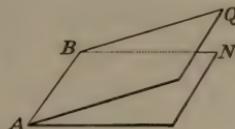


FIG. 2

Thus the size of dihedral angle $N-AB-Q$ is the amount the plane AN must turn on AB as an axis to assume the position AQ . The planes MN and PQ are, of course, understood to be unlimited in extent. Note how the dihedral angle is designated. Which is angle $M-AB-Q$? $M-AB-P$? When two planes meet but do not cross (Fig. 2), the angle may be designated by the intersection only as angle AB .

449. Plane angle of a dihedral angle. Two lines, one in each face of the dihedral angle, perpendicular to the edge at a common point, form a plane angle. Such an angle is the plane angle of the dihedral angle. In Figure 2, if NB and QB are each perpendicular to AB , then $\angle NBQ$ is the plane angle of the dihedral angle.

450. It is evident that all the plane angles of any given dihedral angle are equal, since the sides are respectively parallel, being perpendicular to the same line (§ 446). It is evident also that the dihedral angle *is measured by its plane angle*; that is, *two dihedral angles have the same ratio as their plane angles*.

451. Dihedral angles are classified as *acute, right, obtuse, straight, vertical, adjacent, complementary, supplementary*, etc., according as their plane angles are acute, right, obtuse, straight, etc. Moreover, many of the properties of plane angles are found also in dihedral angles.

452. Perpendicular planes. Two planes that form a right dihedral angle are said to be perpendicular to each other, or in other words, two planes that are perpendicular to each other are said to form a right dihedral angle.

EXERCISES

Ex. 1. Make a sketch to illustrate (1) a right dihedral angle, (2) supplementary dihedral angles, (3) complementary dihedral angles, (4) vertical dihedral angles.

 **Ex. 2.** Can you tell whether or not two planes are parallel by measuring their distance apart in two places? in three?

Ex. 3. How many dihedral angles are there in a cube? Are there as many as there are edges?

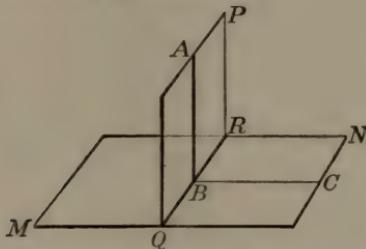
Ex. 4. If a line is drawn in one face of a right dihedral angle \perp to its edge, is it \perp to the other face?

HINT. Form the plane angle of the dihedral angle by drawing a line in the other face \perp to the edge at the same point. Is the given line now \perp to two lines in the second face?

Ex. 5. The cables that hold a derrick rigid are fastened at the top 140 ft. from the ground. They are 220 ft. long. How far from the foot of the derrick (horizontally) are they fastened to the ground?

Proposition 15. Line Perpendicular to a Plane

453. Theorem. *If two planes are perpendicular to each other, a line drawn in one of them perpendicular to the intersection is perpendicular to the other.*



Given plane $PQ \perp$ to plane MN and intersecting MN in QR , and AB in plane $PQ \perp$ to QR at B .

To prove that AB is \perp to MN .

The plan is to draw BC in plane $MN \perp$ to QR and show that AB is \perp to both QR and BC and hence \perp to MN .

Proof. Draw BC in $MN \perp$ to QR at B .

- | | |
|---|--------------------|
| 1. $\angle ABC$ is the plane \angle of dihedral $\angle P-QR-N$. | 1. § 449. |
| 2. But $\angle P-QR-N$ is a rt. dihedral \angle . | 2. By hyp. (§ 450) |
| 3. Hence ABC is a rt. \angle . | 3. § 451. |
| 4. And AB is \perp to BC and QR . | 4. Why? |
| 5. $\therefore AB$ is \perp to MN . | 5. § 419. |

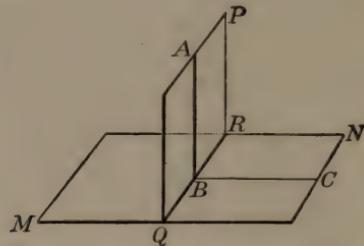
Therefore, if two planes are perpendicular

454. Corollary. *If two planes are perpendicular to each other, a line drawn perpendicular to one of them through any point of the other will lie in the second plane.*

Let AX be any other \perp to MN (see figure above) from any point A in PQ or any point B in QR . Then AX or BY must coincide with AB (§ 429) and therefore lies in PQ .

Proposition 16. Perpendicular Planes

455. Theorem. *If a line is perpendicular to a given plane, every plane that contains this line is perpendicular to that plane.*



Given $AB \perp$ to MN , and PQ any plane containing AB .
To prove that PQ is \perp to MN .

The plan is to draw BC in plane $MN \perp$ to QR and prove that $\angle ABC$ is the plane angle of dihedral angle $P-QR-N$ and that ABC is a rt. \angle .

Proof. From B draw BC in plane $MN \perp$ to QR .

- | | |
|--|-----------|
| 1. AB is \perp to QR , | 1. Why? |
| 2. and BC is \perp to QR . | 2. Why? |
| 3. $\therefore \angle ABC$ is the plane \angle of dihedral $\angle A-QR-C$. | 3. § 449. |
| 4. But AB is \perp to BC . | 4. § 415. |
| That is, $\angle ABC$ is a rt. \angle , | |
| 5. and $A-QR-C$ is a rt. dihedral \angle . | 5. § 450. |
| 6. $\therefore PQ$ is \perp to MN . | 6. § 452. |

Therefore, if a line is perpendicular to a given plane

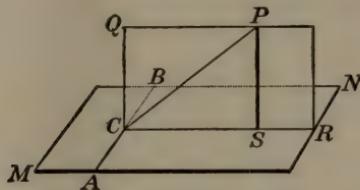
Ex. 1. If a line is parallel to a plane, is a plane parallel to the line parallel to the plane?

Ex. 2. Four parallel planes cut lines AB and CD at A, E, F , B , and C, G, H, D , respectively, so that $AE = 6''$, $EF = 8''$, $FB = 10''$, and $CG = 11''$. Find GH and HD .

Ex. 3. AB lies in plane MN . MN is \perp to plane PQ . Is $AB \perp$ to PQ ?

Proposition 17. Line Perpendicular to Plane

456. Problem. *Draw a line perpendicular to a given plane from a given external point.*



Given plane MN and point P not in MN .

Required to draw a line from $P \perp$ to MN .

The plan is to pass a plane through $P \perp$ to MN and draw the required \perp in this plane. (Can you give the steps in the construction and the proof without reading further?)

Steps in construction.

- | | |
|--|--------------------------------|
| 1. Draw any line AB in MN and pass
a plane QR through $P \perp$ to AB .
2. In QR drop a \perp from P to CR . | 1. § 427.

2. Authority? |
|--|--------------------------------|

Proof.

- | | |
|---|------------------------|
| 1. Plane QR is \perp to MN .
2. $\therefore PS$ is \perp to MN . | 1. § 455.
2. § 453. |
|---|------------------------|

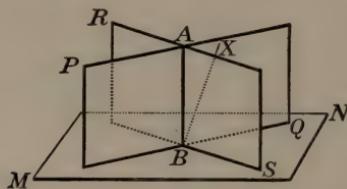
Ex. 1. If lines are drawn from any point within a right dihedral angle perpendicular to the faces, what relation exists between the angle formed by the lines and the plane angle of the dihedral angle?

Ex. 2. Will a plane through P in the figure for § 456 parallel to MN be perpendicular to PS ?

Ex. 3. Is AB in the above diagram perpendicular to the intersection of planes MN and QR ?

Proposition 18. Intersection of Planes

457. Theorem. *If each of two intersecting planes is perpendicular to a third plane, the intersection of the first two is perpendicular to the third.*



Given planes PQ and RS each perpendicular to MN and intersecting each other in AB , B being in MN .

To prove that AB is \perp to MN .

The plan is to show by the indirect method that any line BX that is \perp to MN at B must coincide with AB .

Proof. Assume for the moment that AB is not \perp to MN . Erect a line at B which is \perp to MN . Call this line BX .

- | | |
|--|--|
| 1. BX must lie in PQ and in RS .
2. Hence BX must coincide with AB .
3. But BX is \perp to MN .
4. $\therefore AB$ is \perp to MN . | 1. § 454.
2. Why?
3. By construction.
4. § 429. |
|--|--|

Therefore, if each of two intersecting planes

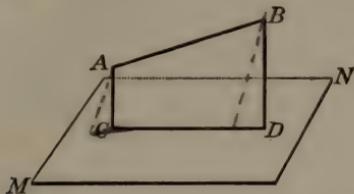
Ex. 1. Can a plane be perpendicular to one of two parallel lines without being perpendicular to the other?

Ex. 2. If a line and a plane are parallel, is a plane perpendicular to the line perpendicular to the plane also?

Ex. 3. If a line and a plane are parallel, is a plane perpendicular to the plane perpendicular to the line also?

Proposition 19. Perpendicular Planes

458. Theorem. *Through a given line not perpendicular to a plane, one and only one plane can be passed perpendicular to the given plane.*



Given a line AB not \perp to plane MN .

To prove that through AB one and only one plane can be passed \perp to MN .

The plan is to draw $AC \perp$ to MN and show that the plane of AB and AC is the only plane through $AB \perp$ to MN . (Can you give the proof without reading further?)

Proof.

- | | |
|---|--|
| 1. Draw $AC \perp$ to MN .
2. Pass a plane through AB and AC .
3. Plane CB is \perp to MN . | 1. § 456.
2. Post. 17.
3. § 455. |
|---|--|

Assume that another plane AX can be drawn through $AB \perp$ to MN .

- | | |
|--|------------------------------|
| 4. AC lies in plane AX .
5. \therefore plane AX coincides with plane CB . | 4. § 454.
5. Post. 17 (3) |
|--|------------------------------|

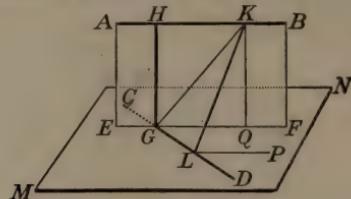
Therefore, through a given line

Ex. 1. If the given line of Proposition 19 is perpendicular to the given plane, can a plane be passed through it perpendicular to the given plane? Can more than one?

Ex. 2. Prove that the shortest line between two parallel planes is perpendicular to them.

Proposition 20. Common Perpendicular to Skew Lines

459. Theorem. *Between any two skew lines there is one and only one common perpendicular.*



Given AB and CD , any two skew lines.

To prove (1) that between AB and CD a common \perp can be drawn and (2) that there is only one such common \perp .

Proof of 1.

- | | |
|--|-----------|
| 1. Through CD pass a plane $MN \parallel$ to AB . | 1. § 440. |
| 2. Through AB pass a plane $BE \perp$ to MN , intersecting it in EF . | 2. § 458. |
| 3. EF is \parallel to AB . | 3. § 439. |
| 4. CD and EF are not \parallel (Why?) and intersect at a point (call it G). In plane BE draw $GH \perp$ to EF . | 4. Why? |
| 5. GH is \perp to AB . | 5. § 112. |
| 6. Now GH is \perp to MN . | 6. § 453. |
| 7. $\therefore GH$ is \perp to CD and is the required common \perp . | 7. § 415. |

Proof of 2.

Assume that there is another common \perp KL . In MN draw $LP \parallel$ to EF .

- | | |
|---|-----------|
| 8. KL is \perp to LP . | 8. § 112. |
| 9. Hence KL is \perp to MN . | 9. § 422. |
| From K drop $KQ \perp$ to EF . | |
| 10. Now we have two \perp s from K to MN , which is impossible. | 10. Why? |

Therefore, between any two skew lines . . .

460. Loci in space. The meaning of a locus is the same in solid geometry as it is in plane geometry. In solid geometry, however, the locus of a point is not confined to a plane. It is the *line or surface, plane or curved, that contains all the points in space that satisfy the conditions imposed upon it, and no other points.*

The *locus* is established, then, by showing:

(1) *that all the points that lie on the locus satisfy the given condition, and*

(2) *that all points that satisfy the given condition lie on the locus.* For example, if you were asked for the locus of all points in space at a given distance from a fixed point, you would establish the locus as a sphere with the given distance for its radius, if you showed (1) that all the points on the sphere are at the given distance from the given point and (2) that all the points at the given distance from the given point lie on the sphere.

Ex. 1. What is the locus of points equidistant from two \parallel planes?

Ex. 2. What is the locus of points in space equidistant from two given points?

Ex. 3. What is the locus of points in space equidistant from all the points of a circle?

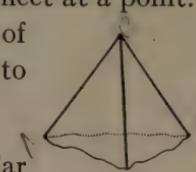
Ex. 4. What is the locus of points equidistant from the three vertices of a triangle?

HINT. What point in the plane of the \triangle is equidistant from the vertices? Now use Exercise 3.

Ex. 5. What is the locus of points equidistant from the sides of a dihedral angle?

HINT. Study the dihedral angle formed by two leaves of a book.

461. Polyhedral angles. A polyhedral angle is the figure formed by three or more planes that meet at a point. The size of the angle depends on the ratio of the portion of angular space it occupies to the entire angular space about the point.

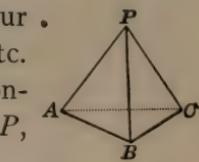


Ex. 1. What portion of the entire angular space about the corner of a rectangular room is occupied by the polyhedral angle formed at the corner by the walls and the ceiling?

The point of intersection of the planes is called the *vertex* of the polyhedral angle, the portion of the planes bounding it are its *faces*, the lines of intersection of the planes are its *edges*, and the dihedral angles between the planes are the *dihedral angles* of the polyhedral angle. The plane angles formed by the edges are called the *face angles* of the polyhedral angle.

A polyhedral angle formed by three planes is a *triangular angle*, one formed by four planes is a *tetrahedral angle*, one formed by five planes is a *pentahedral angle*, etc.

In the accompanying figure there are four trihedral angles: $P-ABC$, $A-BPC$, etc. These are sometimes designated, when no confusion would result, as the trihedral angle P , the trihedral angle A , etc.



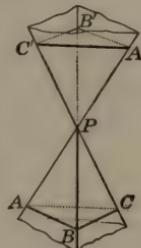
The dihedral angles $B-AP-C$, $A-PB-C$, and $B-PC-A$ are the dihedral angles of the trihedral angle P .

A *convex polyhedral angle* is one that lies wholly on one side of a plane that is passed through any two of its edges. Unless otherwise stated, a polyhedral angle is convex.

Ex. 2. How many trihedral angles has a tetrahedron? How many dihedral angles?

462. Congruent polyhedral angles. If two polyhedral angles can be placed so that their vertices and edges coincide, the polyhedral angles are said to be *congruent*. Two polyhedral angles are congruent if their corresponding parts are equal each to each and they are arranged in the same order.

463. Symmetric polyhedral angles. If the edges of a polyhedral angle be extended through the vertex, another polyhedral angle is formed with the parts of the second respectively equal to the parts of the first, but arranged in the reverse order. The two polyhedral angles are said to be *symmetric*. Thus polyhedral $\triangle P-ABC$ and $P-A'B'C'$ are symmetric polyhedral angles. Except in special cases, symmetric polyhedral angles cannot be superposed.



Ex. 1. Designate the face angles of trihedral angle A in the figure above.

Ex. 2. Name its dihedral angles.

Ex. 3. How many right trihedral angles fill the space about a point? (A *right trihedral angle* is one having three right dihedral angles.)

Ex. 4. If one end of a line segment is fixed and the other is moved around a triangle, is a trihedral angle formed? Always? More than one?

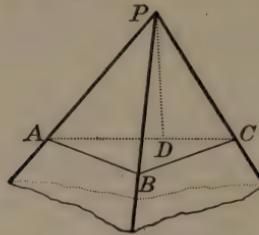
Ex. 5. Does a trihedral angle always have acute angles for its face angles?

Ex. 6. What kind of polyhedral angle is formed at the corners of your schoolroom?

Ex. 7. How does the number of edges in a polyhedral angle compare with the number of faces?

Proposition 21. Face Angles of a Trihedral Angle

464. Theorem. *The sum of any two face angles of any convex trihedral angle is greater than the third face angle.*



Given trihedral $\angle P-ABC$, with APC its greatest face angle.

To prove that $\angle APB + \angle BPC > \angle APC$.

Proof. In plane APC draw PD so that $\angle APD = \angle APB$, and draw through D any line cutting PA and PC , as line ADC , and draw AB and BC making $PB = PD$. In $\triangle APB$ and APD ,

1. $AP = AP$,
2. $PB = PD$,
3. and $\angle APB = \angle APD$.
4. Hence $\triangle APB \cong \triangle APD$,
5. and $AB = AD$.
6. But $AB + BC > AC$.
7. Hence $BC > DC$.

In $\triangle PDC$ and PBC ,

8. $PB = PD$,
 9. and $PC = PC$.
 10. But $BC > DC$.
 11. Hence $\angle BPC > \angle DPC$.
 12. $\therefore \angle APB + \angle BPC > \angle APD + \angle DPC$.
- or $\angle APB + \angle BPC > \angle APC$.

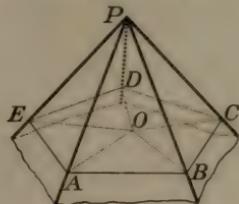
1. Why?
2. By construction.
3. Why?
4. Why?
5. c.p.c.t.e.
6. Post. 3.
7. Ax. 10.

8. Why?
9. Why?
10. Why?
11. § 178.
12. Ax. 10.

Therefore, the sum of any two face angles

Proposition 22. Face Angles of a Polyhedral Angle

465. Theorem. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Given the convex polyhedral angle $P-ABCDE$.

To prove that the sum of face $\angle APB, BPC, CPD, \text{ etc.}$, is less than 4 rt. \angle .

The plan is to prove that the sum of the face \angle at $P <$ the sum of the \angle about any point O in a plane cutting all the edges.

Proof. Pass a plane cutting all the edges of polyhedral $\angle P-ABCDE$ at A, B, C, D, E , as shown. From O , any point in the polygon $ABCDE$, draw $OA, OB, OC, \text{ etc.}$

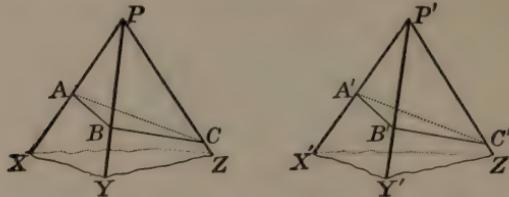
- | | |
|---|---|
| <ol style="list-style-type: none"> 1. The sum of the \angle of the triangles whose vertices are at $P =$ the sum of the \angle of the triangles whose vertices are at O. 2. But $\angle PBA + \angle PBC > \angle ABC$,
 $\angle PCB + \angle PCD > \angle BCD$,
 and $\angle PDC + \angle PDE > \angle CDE$, etc. 3. Hence the sum of the base \angle of the \triangle whose vertices are at P is greater than the sum of the base \angle of the triangles whose vertices are at O. 4. Hence the sum of the vertex \angle at P is less than the sum of the vertex \angle at O. 5. But the sum of the \angle at $O = 4 \text{ rt. } \angle$. 6. \therefore the sum of the \angle at $P < 4 \text{ rt. } \angle$. | <ol style="list-style-type: none"> 1. Each sum
 $= 2 n \text{ rt. } \angle$. 2. § 464. 3. Ax. 10. 4. Ax. 12. 5. Why? 6. Ax. 6. |
|---|---|

Therefore, the sum of the face angles



Proposition 23. Congruent Trihedral Angles

466. Theorem. Two trihedral angles are congruent if the three face angles of one are equal respectively to the three face angles of the other and they are arranged in the same order.



Given trihedral $\angle P-XYZ$ and $P'-X'Y'Z'$, with face $\angle XPY =$ to $\angle X'P'Y'$, $\angle Xpz =$ to $\angle X'P'Z'$, $\angle YPz =$ to $\angle Y'P'Z'$, and arranged in the same order.

To prove trihedral $\angle P-XYZ \cong$ trihedral $\angle P'-X'Y'Z'$.

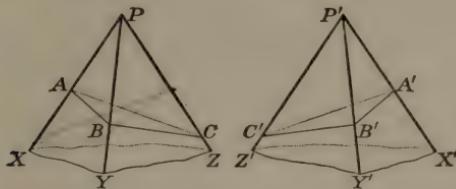
Proof. Lay off $P'A' = PA$ as shown. Pass planes through A and A' \perp to PX and $P'X'$, respectively, cutting the other edges at B , C , B' , and C' , respectively. Draw AB , AC , BC , $A'B'$, $A'C'$, and $B'C'$. In $\triangle APB$ and $A'P'B'$,

- | | |
|--|--|
| <ol style="list-style-type: none"> 1. $\angle A = \angle A'$ and $PA = P'A'$, 2. and $\angle P = \angle P'$. 3. Hence $\triangle APB \cong \triangle A'P'B'$. 4. Hence $AB = A'B'$ and $PB = P'B'$.
Similarly, $\triangle APC \cong \triangle A'P'C'$. 5. Hence $AC = A'C'$ and $PC = P'C'$. 6. But $\angle BPC = \angle B'P'C'$. 7. Hence $\triangle BPC \cong \triangle B'P'C'$. 8. Hence $BC = B'C'$. 9. Hence $\triangle ABC \cong \triangle A'B'C'$. 10. Hence $\angle BAC = \angle B'A'C'$. 11. Hence dih. $\angle PX =$ dih. $\angle P'X'$. 12. \therefore trih. $\angle P-XYZ \cong \angle P'-X'Y'Z'$. | <ol style="list-style-type: none"> 1. By construction. 2. By hypothesis. 3. a.s.a. = a.s.a. 4. c.p.c.t.e. 5. c.p.c.t.e. 6. By hypothesis. 7. s.a.s. = s.a.s. 8. c.p.c.t.e. 9. s.s.s. = s.s.s. 10. c.p.c.t.e. 11. § 450. 12. § 462. |
|--|--|

Similarly, the other two dihedral angles are equal.

467. Corollary. Two trihedral angles are symmetric if the three face angles of one are equal respectively to the three face angles of the other but are arranged in the reverse order.

468. Corollary. If two trihedral angles have the three face angles of one equal respectively to the three face angles of the other, the dihedral angles of one are equal to the dihedral angles of the other.



11

CUMULATIVE REVIEW

Ex. 1. If two planes have a point in common, must they have more than one? *yes*

Ex. 2. Can there be more than one perpendicular from a line to a plane? *yes*

Ex. 3. Must all the perpendiculars of Ex. 2 lie in a plane?

Ex. 4. Can there be more than one perpendicular from a point to a plane? *no*

Ex. 5. If a line is perpendicular to a line of a plane, is it perpendicular to the plane?

Ex. 6. If a line is perpendicular to two lines of a plane, is it perpendicular to the plane?

Ex. 7. If a line is perpendicular to two intersecting lines at their point of intersection, is it perpendicular to the plane of the lines?

Ex. 8. Can a line be perpendicular to each of two planes?

Ex. 9. If two planes are each perpendicular to a third plane, are they parallel? *not necessarily*

Ex. 10. If two lines are each perpendicular to the same plane, are they parallel? *yes*

Ex. 11. If two lines are parallel to the same plane, are they parallel? *No, necessarily*

Ex. 12. If a plane contains one of two parallel lines, is it parallel to the other? Always?

Ex. 13. If a plane is passed through the edge of a dihedral angle, does it make equal dihedral angles with the faces? *No*

Ex. 14. If a plane is passed through the edge of a dihedral angle perpendicular to the plane that bisects the angle, does it make equal angles with the faces? *No*

→ **Ex. 15.** If a line intersects two other lines, do they all lie in the same plane? *No*

→ **Ex. 16.** Is there more than one part to the locus of points equidistant from each of two intersecting planes? Illustrate.

Ex. 17. If a plane is passed perpendicular to a line at its midpoint, are the distances of the ends of the line from any point of the plane equal?

Ex. 18. Lines AB and CD are skew lines. (a) Can a plane be passed through both of them? (b) Can a plane be passed through one of them parallel to the other? (c) Can a plane be passed perpendicular to both of them? (d) Can a plane be passed through one perpendicular to the other?

Ex. 19. If a line and a plane are both perpendicular to the same line, are they parallel? *Yes*

Ex. 20. If a portion of a plane is moved in any other manner than sliding, what kind of geometric figure is generated?

Ex. 21. Do points equidistant from two parallel planes all lie in a plane?

Ex. 22. Will three equal wires, fastened at the same height on a smokestack, hold it rigid? What principle is involved here?

Ex. 23. Why does a photographer support his camera on a tripod?

Ex. 24. Can three planes bound a solid? *No*

Ex. 25. Can you use a protractor to measure a dihedral angle? Directly? Indirectly?

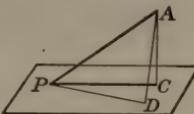
No *No* *No*

Ex. 26. If the face angles of a trihedral angle are equal, are the dihedral angles all equal?

> **Ex. 27.** If four lines, no three of which lie in the same plane, meet at a point, how many different planes do they determine? Illustrate.

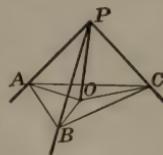
Ex. 28. If several planes pass through a common line and are each perpendicular to a given plane, is the common line of the several planes perpendicular to the given plane?

Ex. 29. Can you prove that the acute angle between a straight line and its projection on a plane is the least angle that it can make with any line drawn in the plane through its foot?



HINT. Take $PD =$ to PC and draw AD . Compare $\triangle APC$ and APD .

Ex. 30. Having given three straight lines meeting in a point, P , but not lying in the same plane, can you draw through P a line making equal angles with the three given lines?



HINT. Pass a plane cutting the three lines so that $PA = PB = PC$. Can you find a point O in the plane of A, B , and C such that $\triangle POA \cong \triangle POB \cong \triangle POC$? Would O be equidistant from A, B , and C ?

Ex. 31. Can you find the locus of points equidistant from the faces of a trihedral angle?

Ex. 32. Do you think that the planes bisecting the dihedral angles of a trihedral angle all meet in a line?

HINT. (See figure above.) Are all points that are common to the bisector of dihedral $\angle PA$ and dihedral $\angle PB$ equidistant from the sides of dihedral $\angle PC$?

Ex. 33. Can you prove that the shortest distance between two skew lines is their common perpendicular?

HINT. Pass a plane through one line \parallel to the other. What is the shortest distance from the plane to the second line?

BOOK SEVEN

POLYHEDRONS, PRISMS, CYLINDERS, AND CONES



POLYHEDRONS

469. A **polyhedron** is a solid bounded by planes. The bounding planes are its *faces*; the lines of intersection of the faces are its *edges*; and the points of intersection of its edges are its *vertices*.

470. A *diagonal* of a polyhedron is a line segment joining two vertices that do not lie in the same face.

471. The **volume** of a solid is the number of cubic units it contains.

472. **Equivalent solids** are solids having equal volumes.

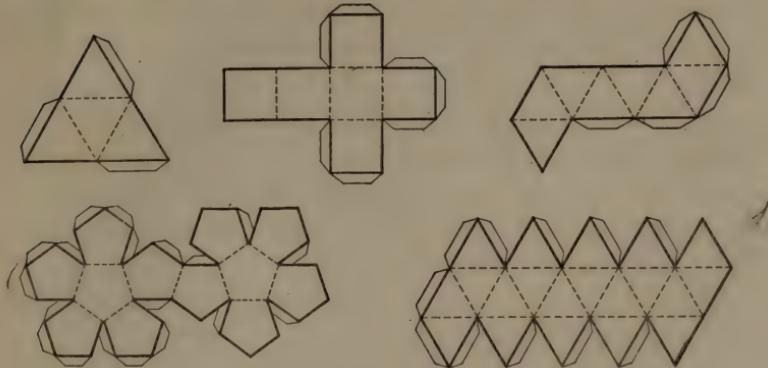
473. **Congruent solids.** Solids that can be made to coincide are *congruent*. Two polyhedrons that have their corresponding parts equal and arranged in the same order are congruent.

474. There are many types of polyhedrons, ranging in complexity from the cube to solids of very complicated forms, and include *prisms*, *pyramids*, and *irregular solids* with plane surfaces.

Polyhedrons are classified in accordance with the number of planes that bound them; thus a *tetrahedron* is a polyhedron of four faces, a *pentahedron* is one of five faces, etc.

475. A regular polyhedron has equal polyhedral angles. Its faces are congruent regular polygons.

476. Copy on cardboard the following patterns, cut them out, fold along the dashed lines and paste down the flaps.



477. There cannot be more than these five regular polyhedrons. At least three planes are necessary to form a polyhedral angle, and the sum of the face angles of a polyhedral angle must be less than 360° (§ 465). Beginning with equilateral triangles, we see that it is possible to place together at one vertex either three, four, or five 60° angles,

$$\text{since } 3 \times 60^\circ = 180^\circ, \text{ which is less than } 360^\circ,$$

$$4 \times 60^\circ = 240^\circ, \text{ which is less than } 360^\circ,$$

$$5 \times 60^\circ = 300^\circ, \text{ which is less than } 360^\circ.$$

We can bring together at one vertex three squares, since

$$3 \times 90^\circ = 270^\circ, \text{ which is less than } 360^\circ.$$

We can also bring together three regular pentagons, since

$$3 \times 108^\circ = 324^\circ, \text{ which is less than } 360^\circ.$$

Regular polygons of more than five sides cannot be used; since the sum of three angles would not be less than 360° .

PRISMS

478. A **prism** is the solid generated (passed through) by a polygon moving so that its sides generate parallelograms.

The congruent polygons constituting the initial and terminal positions of the generating polygon (the two ends) are called its *bases*. The bases, of course, lie in parallel planes.

The parallelograms generated by the moving sides of the polygon are called its *lateral faces*.

The parallel edges generated by the moving vertices of the polygon are called its *lateral edges*. The lateral edges of a prism are, of course, equal.

The sum of the areas of the lateral faces is called the *lateral area* of the prism.

The perpendicular distance between the planes of the bases of a prism is called the *altitude* of the prism.

479. Right prism. A prism whose lateral edges are perpendicular to its bases is called a *right prism*. The altitude of a right prism is equal to the length of a lateral edge.

480. Oblique prism. A prism whose edges are oblique to its bases is called an *oblique prism*.

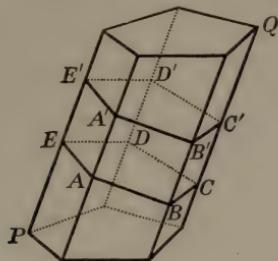
481. Prisms classified. Prisms are triangular, quadrangular, square, regular, etc., according as their bases are triangular, quadrangular, square, regular, etc. A *right regular prism* is both right and regular.

482. A right section of a prism is the polygon formed by passing a plane cutting all the lateral edges at right angles.

483. A truncated prism is the part of a prism included between one base and a plane oblique to the base which cuts all the lateral edges.

Proposition 1. Parallel Sections of a Prism

484. Theorem. *The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.*



Given prism PQ cut by parallel planes cutting all edges in A, B, C, D, E , and A', B', C', D', E' , respectively.

To prove that polygon $ABCDE$ is congruent to polygon $A'B'C'D'E'$.

The plan is to show that the polygons $ABCDE$ and $A'B'C'D'E'$ are mutually equiangular and mutually equilateral. (See if you can give the proof without reading further.)

Proof.

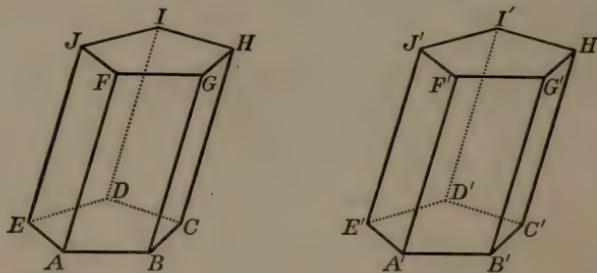
- | | |
|---|-----------|
| 1. $ABB'A'$ is a parallelogram. | 1. § 478. |
| 2. Hence $AB = A'B'$.
Likewise $BC = B'C'$, $CD = C'D'$, $DE = D'E'$, etc. | 2. § 145. |
| 3. $\angle ABC = \angle A'B'C'$, $\angle BCD = \angle B'C'D'$, etc.
Hence the polygons are mutually equiangular
and mutually equilateral. | 3. § 446. |
| 4. \therefore polygon $ABCDE \cong$ polygon $A'B'C'D'E'$. | 4. § 56. |

Therefore, the sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.

485. Corollary. *Every section of a prism made by a plane parallel to the base is congruent to the base.*

Proposition 2. Congruent Prisms

486. Theorem. *Two prisms are congruent if the three faces which include a trihedral angle of one are congruent to the three faces which include a trihedral angle of the other and they are similarly placed.*



Given prisms AI and $A'I'$, in which face $AD \cong$ face $A'D'$, face $AG \cong$ face $A'G'$, face $AJ \cong$ face $A'J'$, and they are similarly placed.

To prove that prism $AI \cong$ prism $A'I'$.

The plan is to prove that trihedral $\angle A \cong$ trihedral $\angle A'$, to superpose the prisms so that trihedral $\angle A$ will coincide with trihedral $\angle A'$, and show that the prisms coincide.

Proof.

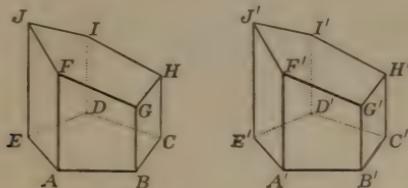
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|--|---|
| <ol style="list-style-type: none"> 1. $\angle EAB = \angle E'A'B'$, $\angle FAB = \angle F'A'B'$, and $\angle FAE = \angle F'A'E'$. 2. Hence trihedral $\angle A \cong$ trihedral $\angle A'$. 3. AD will coincide with $A'D'$. Likewise face AG will coincide with face $A'G'$, and face AJ with face $A'J'$. | <ol style="list-style-type: none"> 1. Cor. parts of cong. polygons are equal. 2. § 466. 3. They are congruent by hypothesis.
(§ 56.) |
|--|---|

- | | |
|--|--|
| 4. Hence the plane of GFJ coincides with plane of $G'F'J'$.
5. But face $FI \cong$ face AD and face $F'I' \cong$ face $A'D'$.
6. Hence face $FI \cong$ face $F'I'$.
7. Hence FI coincides with $F'I'$.
8. $\therefore CH$ coincides with $C'H'$ and DI with $D'I'$.
That is, the prisms coincide throughout. | 4. Post. 17.
5. § 478.
6. Ax. 6.
7. Why?
8. Why? |
|--|--|

Therefore, two prisms are congruent if the three faces which include a trihedral angle of one

487. Corollary. *Two right prisms are congruent if they have congruent bases and equal altitudes.*

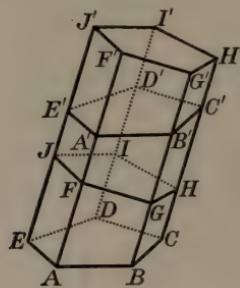
488. Corollary. *Two truncated prisms are congruent if the three faces which include a trihedral angle of one are congruent respectively to the three faces which include a trihedral angle of the other, and they are similarly placed.*



Let the faces AD, AG, AJ be congruent with $A'D', A'G', A'J'$. Hence trihedral $\angle A \cong$ trihedral $\angle A'$ (§ 466). Place the truncated prisms so that trihedral $\angle A$ coincides with trihedral $\angle A'$ (§ 56) with FG coinciding with $F'G'$ and FJ with $F'J'$. Plane FI will coincide with plane $F'I'$ (Post. 17). But CH and DI are parallel to AF . Hence CH and $C'H'$ will coincide (Post. 15), and H will fall at H' (Post. 19). Likewise I will fall at I' . Therefore the prisms coincide throughout and are congruent (§ 56).

Proposition 3. Oblique Prism

489. Theorem. *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Given oblique prism AD' and the right prism FI' whose base FI is a right section of AD' and whose altitude is equal to lateral edge AA' of prism AD' .

To prove that prism AD' is equivalent to prism FI' .

The plan is to prove that truncated prism $AI \cong$ truncated prism $A'I'$ and that $FD' + AI = FD' + A'I'$.

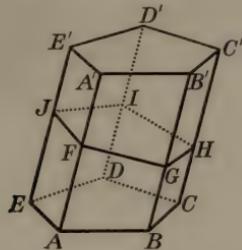
Proof.

- | | |
|---|-----------|
| 1. $AF = A'F'$, $BG = B'G'$, $CH = C'H'$, etc. | 1. Ax. 2. |
| 2. The angles of face AG = the angles of face $A'G'$. | 2. Why? |
| 3. $AB = A'B'$ and $FG = F'G'$. | 3. Why? |
| 4. Hence face $AG \cong$ face $A'G'$. | 4. Why? |
| Likewise face $AJ \cong$ face $A'J'$. | |
| 5. But face $AD \cong$ face $A'D'$. | 5. Why? |
| 6. Hence prism $AI \cong$ prism $A'I'$. | 6. § 488. |
| 7. Hence prism $FD' +$ prism $AI =$ prism $FD' +$ prism $A'I'$; that is, prism $AD' =$ prism FI' . | 7. Why? |

Therefore, an oblique prism

Proposition 4. Lateral Area of a Prism

490. Theorem. *The lateral area of a prism is equal to the product of a lateral edge by the perimeter of a right section.*



Given prism AD' , with the lateral edges AA' , BB' , etc., each equal to e , the perimeter of the right section, FI , equal to p , and a lateral area, L .

To prove that $L = ep$.

The plan is to show that the perimeter of the right section is the sum of the altitudes of the parallelograms constituting the lateral faces.

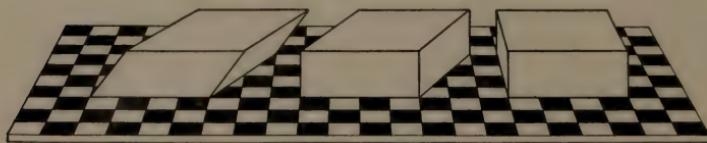
Proof. Lettering the figure as shown,

- | | |
|--|-------------------|
| 1. $AA' = BB' = CC' = DD' = EE' = e$. | 1. § 478. |
| 2. FG is \perp to AA' , GH is \perp to BB' , etc. | 2. § 482. |
| 3. Hence $\square AB' = e \times FG$, | 3. Why? |
| $\square BC' = e \times GH$, etc. | |
| 4. Hence $\square AB' + \square BC' + \square CD' + \square DA' = e(FG + GH + HI + IJ + JF)$. | 4. Ax. 1. |
| 5. Now $FG + GH + HI + IJ + JF = p$, | 5. By hypothesis. |
| 6. and $\square AB' + \square BC' + \square CD' + \square DA' = L$. | 6. By hypothesis. |
| 7. $\therefore L = ep$. | 7. Ax. 6. |

Therefore, the lateral area of a prism

Ex. 1. In § 490 assume FI to be a regular pentagon. Measure FG and AA' . Assuming an inch to represent 16 feet, compute the lateral area of prism AD' .

PARALLELEPIPEDS



491. A **parallelepiped** is a prism whose bases are parallelograms. (See the left-hand solid above.)

492. A **right parallelepiped** is one which has at least one pair of bases to which the corresponding lateral edges are perpendicular. (See the middle solid above.)

493. A **right rectangular parallelepiped** is a right parallelepiped whose bases are rectangles.¹ (See the right-hand solid above.) Obviously all six faces of a right rectangular parallelepiped are rectangles.

494. A **cube** is a rectangular parallelepiped whose faces are all squares.

495. Dimensions. The lengths of the three edges of a right rectangular parallelepiped that meet at any vertex are called its *dimensions*.

496. The **volume** of a right rectangular parallelepiped (number of units) is the product of the number of units of its dimensions. This law is abbreviated as follows: $V = lhw$.

497. Fundamental Principle. *The volume of a rectangular parallelepiped is the product of its base and its altitude.*

498. Corollary. *The volume of a cube is the cube of one of its dimensions.*

¹A right rectangular parallelepiped is sometimes called merely a rectangular parallelepiped. However, to be consistent, this latter term should be used to indicate *any* parallelepiped having a rectangular base.

499. Diagonal quadrilaterals. The quadrilateral formed by a plane intersecting any two non-adjacent parallel edges of a parallelepiped is called a *diagonal quadrilateral*. There are six of these for a given parallelepiped, one for each pair of non-adjacent edges.

EXERCISES

Ex. 1. Is a prism a polyhedron? Is a parallelepiped a polyhedron?

Ex. 2. Is a polyhedron a prism? Is a parallelepiped a prism?

Ex. 3. Are the lateral edges of a prism equal? Are the lateral edges of a parallelepiped equal?

Ex. 4. Are the lateral faces of a prism equal? Are the bases?

Ex. 5. What is the least number of faces that a polyhedron may have? the largest?

Ex. 6. What is the least number of faces that a prism may have? the largest?

Ex. 7. Are the faces of a parallelepiped rectangles? parallelograms?

Ex. 8. Is a right parallelepiped a rectangular parallelepiped?

Ex. 9. Is a rectangular parallelepiped a right parallelepiped?

Ex. 10. Are the opposite faces of a rectangular parallelepiped congruent?

Ex. 11. Is a square¹ right prism a rectangular parallelepiped?

Ex. 12. Is a section of a tetrahedron by a plane parallel to two non-adjacent edges a parallelogram?

Ex. 13. Is the right section of a square prism a square?

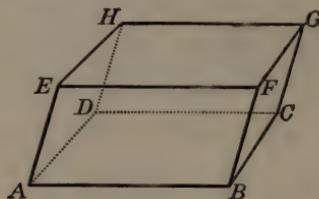
Ex. 14. Is the right section of a square prism a parallelogram?

Ex. 15. Prove that the volumes of two right rectangular parallelepipeds are to each other (1) as the products of their dimensions, (2) as their altitudes, if their bases are equal, (3) as their bases, if their altitudes are equal.

¹ See § 481.

Proposition 5. Congruent Faces

500. Theorem. *The opposite faces of a parallelepiped are congruent and parallel.*



Given the parallelepiped AG having as bases $\square ABCD$ and $EFGH$.

To prove that the opposite faces are congruent and parallel.

The plan is to show that the corresponding sides of opposite lateral faces are equal and parallel. (Can you prove it?)

Proof.

- | | |
|--|---|
| 1. $\square ABCD$ is \cong and \parallel to $\square EFGH$.
2. Hence in $\square ABFE$ and $DCGH$,
$AB = DC = EF = GH$,
3. and $AE = BF = CG = DH$.
4. Now AB is \parallel to DC ,
5. and AE is \parallel to DH .
6. Hence $\angle A = \angle D$, $\angle B = \angle C$,
etc.
7. Hence $\square ABFE \cong \square DCGH$,
8. and these \square are \parallel . | 1. By definition of prism.
2. § 56.

3. § 478.
4. $ABCD$ is a \square . (§ 49.)
5. § 478.
6. § 446.

7. Why?
8. § 446. |
|--|---|

Similarly it may be shown that $\square AEHD \cong$ and \parallel to $\square BFGC$.

Ex. 1. Can the altitude of an oblique prism be equal to a lateral edge?

Ex. 2. If from any point within a parallelepiped perpendiculars are drawn to the faces, is the sum of these perpendiculars equal to the sum of three adjacent edges of the parallelepiped?

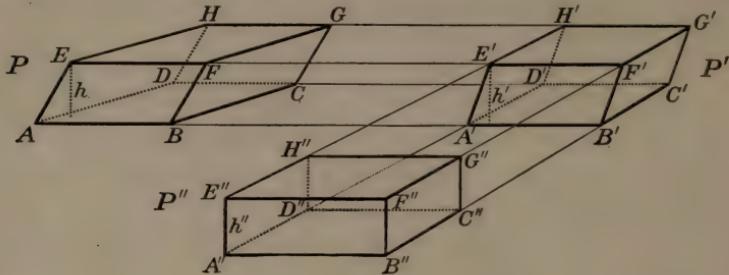
501. Corollary. *Any section of a parallelepiped made by a plane cutting four parallel edges is a parallelogram.*

Since the opposite faces are parallel planes, their intersectings with the cutting plane are parallel. The figure formed, therefore, has two pairs of parallel sides.

- Ex. 1.** Are the lateral edges of a parallelepiped parallel? *Ans. No*
- Ex. 2.** Are the diagonals of a right parallelepiped equal? *Ans. No*
- Ex. 3.** Are the diagonals of a rectangular parallelepiped equal?
- Ex. 4.** Are the diagonal quadrilaterals (§ 499) of a right parallelepiped congruent?
- Ex. 5.** Are all the diagonal quadrilaterals of a right parallelepiped rectangles?
- Ex. 6.** Make a sketch of a parallelepiped having two unequal dihedral quadrilaterals.
- Ex. 7.** Do any two diagonals of a parallelepiped bisect each other?
- Ex. 8.** If the length of one of the diagonals of a cube is known, can the edge be found? the area? the volume?
- Ex. 9.** Are all the faces of a rectangular parallelepiped congruent? *Ans. No*
- Ex. 10.** If one of the edges of a parallelepiped is perpendicular to the base, is it a right parallelepiped?
- Ex. 11.** How many pairs of opposite faces has a parallelepiped?
- Ex. 12.** Are the opposite trihedral angles of a parallelepiped congruent? *Ans. No*
- Ex. 13.** The right section of a prism is a regular hexagon whose side is 10 in. The lateral edge of the prism is 12 in. What is the lateral area?
- Ex. 14.** Can you find the total area of the prism of Exercise 13?

Proposition 6. Volume of a Parallelepiped

502. Theorem. *The volume of any parallelepiped is equal to the product of the base and the altitude.*



Given any parallelepiped P , with base b and altitude h .
To prove that the volume of $P = bh$.

The plan is to prove that parallelepiped $P =$ rectangular parallelepiped $P' =$ right rectangular parallelepiped $P'' = bh$.

Proof.

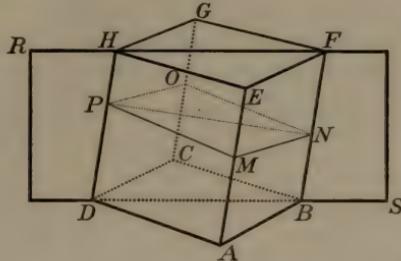
- | | |
|---|------------------|
| 1. Extend edges AB , DC , EF , and HG , and on the extension of AB lay off $A'B' =$ to AB . Through A' and B' pass planes \perp to $A'B'$ making right ¹ prism P' with base b' , (i.e., $A'B'C'D'$) and altitude h' . | 1. § 442, § 479. |
| 2. Extend edges $D'A'$, $C'B'$, $G'F'$, and $H'E'$, and on the extension lay off $C''B'' =$ to $C'B'$ and pass planes through C'' and B'' \perp to $C''B''$ forming rt. rect. parallelepiped P'' , with base b'' and altitude h'' . | 2. § 442, § 493. |
| 3. Now we have $b = b' = b''$, | 3. Why? |
| 4. and $h = h' = h''$. | 4. Why? |
| 5. Also, $P = P' = P''$. | 5. § 489. |
| 6. But $P'' = b''h''$. | 6. § 497. |
| 7. $\therefore P = bh$. | 7. Ax. 5. |

Therefore, the volume of any parallelepiped

¹ P' is a right prism with respect to base $A'B'F'E'$.

Proposition 7. Diagonal Plane

503. Theorem. *The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equivalent triangular prisms.*



Given parallelepiped AG , with RS a plane through HD and BF , two diagonally opposite edges.

To prove that prism $ABD-EFH =$ prism $BCD-FGH$.

The plan is to prove that prisms $ABD-EFH$ and $BCD-FGH$ are equivalent to congruent right prisms.

Proof. Construct right section $MNOP$. Let it intersect RS in PN .

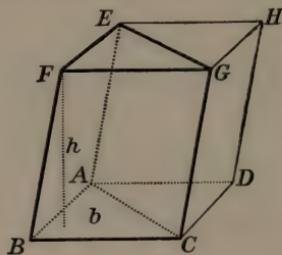
- | | |
|---|-----------|
| 1. Prism $ABD-EFH$ is equivalent to a right prism whose base is MNP and whose altitude is lateral edge BF . | 1. § 489. |
| 2. Also prism $BCD-FGH$ is equivalent to a right prism whose base is NOP and whose altitude is the lateral edge, BF . | 2. Why? |
| 3. But $MNOP$ is a parallelogram. | 3. § 501. |
| 4. Hence $\triangle MNP \cong \triangle NOP$. | 4. Why? |
| 5. Hence the two right prisms are congruent. | 5. § 487. |
| 6. \therefore prism $ABD-EFH =$ prism $BCD-FGH$. | 6. Ax. 5. |

Therefore, the plane passed through . . .

Ex. 1. What are the total area and volume of a cube whose diagonal is d ?

Proposition 8. Volume of Triangular Prism

504. Theorem. *The volume of a triangular prism is equal to the product of the base and the altitude.*



Given any triangular prism $ABC-EFG$ with base b and altitude h .

To prove that the volume of $ABC-EFG = bh$.

The plan is to prove that the triangular prism $ABC-EFG$ is half of a parallelepiped with the same altitude but with a base BD double that of the triangular prism. (Can you prove it?)

Proof. Draw $AD \parallel BC$, $CD \parallel BA$, $EH \parallel FG$, and $GH \parallel FE$, as shown; draw DH .

- | | |
|--|--|
| 1. FD is a parallelepiped with altitude h .
2. The base of $FD = 2 b$.
3. Now the volume of $FD = 2 bh$.
4. Hence the volume of $ABC-EFG = \frac{2 bh}{2} = bh$. | 1. Why?
2. Why?
3. § 502.
4. § 503. |
|--|--|

Therefore, the volume of a triangular prism . . .

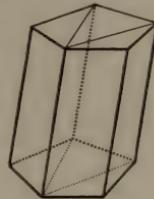
Ex. 1. The base of a triangular prism is an equilateral triangle, each of whose sides is 20 ft. The altitude is 16 ft. What is its volume?

Ex. 2. Can you find the lateral area of the prism of Ex. 1?

Ex. 3. A stick of timber was shipped to Alaska as the base for a stationary engine. It was 80 ft. long, 54 in. wide, and 30 in. thick. How many cubic feet were there in it?

505. Corollary. *The volume of any prism equals the product of the base and the altitude.*

The prism may be divided into triangular prisms by passing diagonal planes through any edge and the non-adjacent edges. The volume of each of these triangular prisms is the product of its base and the common altitude. (§ 504.) The sum of the bases of these triangular prisms is the base of the given prism. (Why?) Hence the truth of the corollary.



Ex. 1. If the right section of a prism is parallel to the base, is it a right prism?

Ex. 2. If the three dimensions of a rectangular parallelepiped are known, can the length of its diagonal be found?

Ex. 3. If you know the length of the three edges meeting at one of the vertices of a parallelepiped, can you find the length of the diagonals?

506. A **pyramid** is a polyhedron formed by joining each of the vertices of a polygon, called the *base*, to a point, called the *vertex*, not in the plane of the base. The triangular planes that meet at the vertex are called the *lateral faces*. The perpendicular distance from the vertex to the base is called the *altitude* of the pyramid.

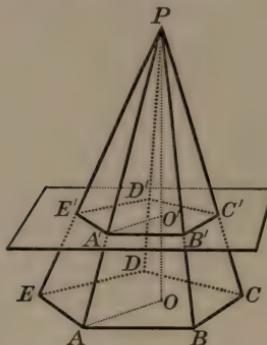
507. Pyramids are classified according to the character of the base, as *triangular*, *quadrangular*, *square*, etc.

508. A **regular pyramid** is one whose base is a regular polygon. A *right regular pyramid* is a regular pyramid whose altitude meets the base at the center.

509. Properties of a right regular pyramid. The lateral edges of a right regular pyramid are all equal, and therefore the faces are congruent isosceles triangles. The altitude of any one of the triangular faces is called the *slant height*.

Proposition 9. Section of a Pyramid

510. Theorem. *If a pyramid is cut by a plane parallel to the base, (1) the lateral edges and the altitude are divided proportionally, (2) the section is a polygon similar to the base, and (3) the area of the section is to the area of the base as the square of the distance of the plane from the vertex is to the square of the altitude of the pyramid.*



Given pyramid $P-ABCDE$, with a section $A'D'E' \parallel$ to base AD cutting altitude PO at O' .

To prove (1) that $\frac{PA'}{A'A} = \frac{PB'}{B'B} = \frac{PC'}{C'C} = \frac{PD'}{D'D} = \frac{PE'}{E'E} = \frac{PO'}{O'O}$, (2) that $A'D'E' \sim AD$, and (3) that $\frac{\text{Area } A'D'E'}{\text{Area } AD} = \frac{\overline{PO'}^2}{\overline{PO}^2}$.

The plan is to show (1) that the sides of polygons AD and $A'D'E'$ are respectively parallel and that the segments of the lateral edges and altitude are therefore divided proportionally, (2) that the sides of section $A'D'E'$ and base AD are therefore respectively proportional and the angles respectively equal, and (3) that the area of the section is to the area of the base as $\overline{A'B'}^2$ is to \overline{AB}^2 .

See if you can prove this proposition without referring to the proof on the next page.

Proof of 1.

1. $A'O'$ is \parallel to AO and $A'B'$ is \parallel to AB , $B'C'$ is \parallel to BC , $C'D'$ is \parallel to CD , etc. 1. § 444.
2. Hence $\frac{PO'}{O'O} = \frac{PA'}{A'A} = \frac{PB'}{B'B} = \frac{PC'}{C'C}$, etc. 2. § 291.

Proof of 2.

3. $\triangle PO'A' \sim \triangle POA$, $\triangle PA'B' \sim \triangle PAB$,
 $\triangle PB'C' \sim \triangle PBC$, etc. 3. § 311.
4. Hence $\frac{O'A'}{OA} = \frac{PA'}{PA} = \frac{A'B'}{AB} = \frac{PB'}{PB} = \frac{B'C'}{BC}$
 $= \frac{PC'}{PC} = \frac{C'D'}{CD}$, etc. 4. § 304.
5. Also $\angle E'A'B' = \angle EAB$, $\angle A'B'C' = \angle ABC$, etc. 5. § 446.
6. Hence polygon $A'D' \sim$ polygon AD . 6. § 304.

Proof of 3.

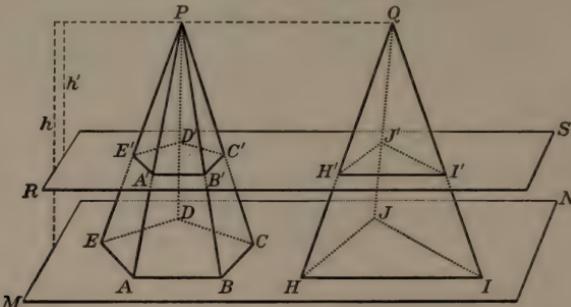
7. $\frac{\text{Area of polygon } A'D'}{\text{Area of polygon } AD} = \frac{\overline{A'B'}^2}{\overline{AB}^2} = \left(\frac{A'B'}{AB}\right)^2$ 7. § 339.
8. But $\frac{A'B'}{AB} = \frac{PA'}{PA} = \frac{PO'}{PO}$. 8. See State-
ment 2.
9. $\therefore \frac{\text{Area } A'D'}{\text{Area } AD} = \left(\frac{PO'}{PO}\right)^2 = \frac{\overline{PO'}^2}{\overline{PO}^2}$ 9. Ax. 15
and Ax. 6.

Ex. 1. In the figure for § 510, if PO equals 20 inches and PO' equals 12 inches, what would be the ratio of the area of AD to that of $A'D'$?

Ex. 2. Can the area of the base of a pyramid be equal to its lateral area?

Ex. 3. The side of the base and the altitude of a right regular hexagonal pyramid are 10 in. and 12 in., respectively. A plane is passed through it parallel to the base 8 in. from the vertex. Find the area of the base, and the side and area of the section.

511. Corollary. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the bases at equal distances from the vertices are equivalent.*



Given pyramids P and Q having equal altitudes h , and equivalent bases $ABCDE$ and HIJ in plane MN , with sections $A'B'C'D'E'$ and $H'I'J'$ made by plane $RS \parallel MN$ at a distance h' from the vertices.

To prove that $A'B'C'D'E'$ is equivalent to $H'I'J'$.

Proof.

$$\frac{A'B'C'D'E'}{ABCDE} = \frac{h'^2}{h^2} \quad \text{and} \quad \frac{H'I'J'}{HIJ} = \frac{h'^2}{h^2}. \quad (\S\ 510.) \quad \text{Hence}$$

$$\frac{A'B'C'D'E'}{ABCDE} = \frac{H'I'J'}{HIJ}. \quad (\text{Ax. 5.}) \quad \text{But} \quad ABCDE = HIJ.$$

(By hypothesis.) $\therefore A'B'C'D'E' = H'I'J'.$ (Ax. 3.)

Ex. 1. In the figure for § 511, does $\frac{PE'}{PE} = \frac{QH'}{QH} = \frac{h'}{h}$?

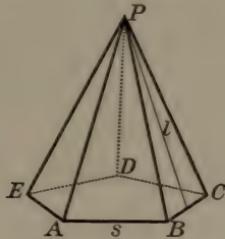
Ex. 2. Two prisms have bases whose ratio is 2 to 3, and altitudes whose ratio is 1 to 2. What is the ratio of the volumes of the two prisms?

Ex. 3. The volume of a right rectangular parallelepiped is 648 cu. in. The edges have the ratio 2 to 3 to 4. Find each edge.

Ex. 4. The surface of a cube is S . What is its edge? its volume?

Proposition 10. Lateral Area of a Pyramid

512. Theorem. *The lateral area of a right regular pyramid is equal to half the product of the slant height and the perimeter of the base.*



Given the right regular pyramid P -ABCDE of n lateral faces, with lateral area A , perimeter of base p , and slant height l , side of base s .

To prove that $A = \frac{1}{2}lp$.

Proof.

- | | |
|---|--|
| 1. The area of lateral face $PAB = \frac{1}{2}l \times AB$.
2. The lateral faces are congruent.
3. Hence the lateral area
$= \frac{1}{2}l \times AB + \frac{1}{2}l \times BC$, etc.,
$= \frac{1}{2}l(AB + BC + CD + \text{etc.})$.
4. \therefore Lateral area $= \frac{1}{2}lp$. | 1. § 271.
2. § 509.
3. Ax. 1.
4. Ax. 6. |
|---|--|

513. A **frustum** of a pyramid is the portion of a pyramid between its base and a plane parallel to its base.

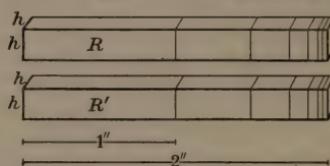
514. The **altitude** of a frustum of a pyramid is the perpendicular distance between its bases.

515. The **slant height** of the frustum of a right regular pyramid is the altitude of one of the congruent trapezoids of its lateral surface.

516. Corollary. *The lateral area of a frustum of a right regular pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height of the frustum.*

*infinitesimal
calculus
series*

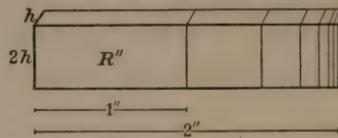
517. **The principles of limits.** The student will recall the discussion of limits in § 387 of *Modern Plane Geometry*. Suppose we have two right rectangular prisms, R and R' ,



each $1''$ long and of the same cross-section (h^2). Suppose we increase the length of each by $\frac{1}{2}''$, then by $\frac{1}{4}''$, then by $\frac{1}{8}''$, then by $\frac{1}{16}''$, etc. What does the length of the two prisms lack of being $2''$ after the first increase? after the second? the third? the fourth? By continuing in this way can we make this difference as small as we like? What, then, is the limit of the length of R if the increasing is carried on indefinitely? What is the limit of the length of R' ?

1. *If while approaching their respective limits two variables are always equal, their limits are equal.*

Let us now suppose we have a right rectangular prism, R'' , $1''$ long, but with a height of $2h$ and width of h . It is,

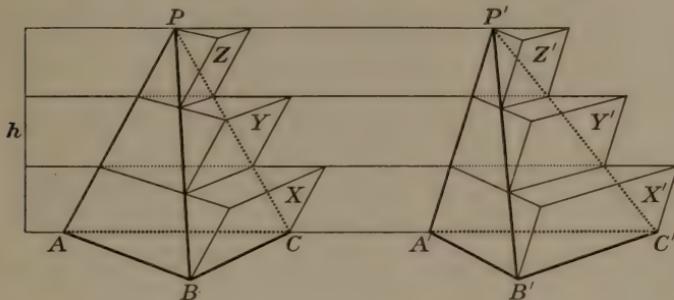


therefore, just twice as large as prism R . Does $R'' = 2R$ when the length of each is $1\frac{1}{2}''$? when the length of each is $1\frac{3}{4}''$? $1\frac{7}{8}''$? $1\frac{15}{16}''$? $1\frac{31}{32}''$? What is the limit of the volume of solid R'' when the increasing has been carried on indefinitely? Is it $2 \times$ the volume of solid R ?

2. *If a variable v approaches a limit l, and if c is a constant, then cv approaches the limit cl.*

Proposition 11. Equivalent Pyramids

518. Theorem. *Two triangular pyramids having equal altitudes and equivalent bases are equivalent.*



Given triangular pyramids $P-ABC$ and $P'-A'B'C'$, with equal altitudes h , and equivalent bases ABC and $A'B'C'$, and volumes V and V' .

To prove that $V = V'$.

Proof. Place the pyramids so that their bases lie in a common plane. Divide h into any number of equal parts and pass planes through the points of division \parallel to the plane of the bases, making sections of the pyramids. Upon the base of each pyramid and upon each section as a base construct a prism, as X , X' , Y , Y' , Z , and Z' .

- | | |
|--|------------------|
| 1. The base of X = the base of X' . | 1. Given. |
| 2. Y and Y' and Z and Z' have equal bases. | 2. § 511. |
| 3. The altitudes are equal. | 3. By constr. |
| 4. Hence $X = X'$, $Y = Y'$, $Z = Z'$. | 4. § 504, Ax. 5. |
| 5. Hence $X + Y + Z = X' + Y' + Z'$. | 5. Why? |

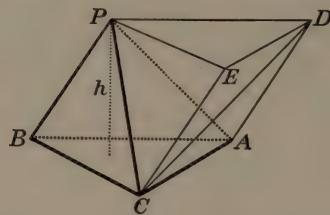
Let the number of divisions of h be continuously increased. Then the sum of the prisms of each pyramid approaches the volume of the pyramid; that is, $X + Y + Z + \text{etc.} \longrightarrow V$ and $X' + Y' + Z' + \text{etc.} \longrightarrow V'$.

6. $\therefore V = V'$.

6. § 517, 1.

Proposition 10. Volume of Triangular Pyramid

519. Theorem. *The volume of a triangular pyramid equals one third the product of the base and the altitude.*



Given triangular pyramid $P-ABC$, of volume V , base b , and altitude h .

To prove that $V = \frac{1}{3}bh$.

The plan is to construct a triangular prism with altitude h on the base of the pyramid, and show that this prism consists of three equivalent pyramids of which the given pyramid is one.

Proof. Complete $\square ABPD$ and $PBCE$. Draw DE .

- | | |
|--|-----------|
| 1. Then $ABC-DPE$ is a prism with base b and altitude h . Let V' represent the volume. | 1. § 478. |
| 2. $V' = bh$. | 2. § 505. |
| If from the prism $ABC-DPE$ the pyramid $P-ABC$ is removed, $P-ADEC$ will remain. | |
| 3. But $ADEC$ is a \square . | 3. Why? |
| Draw DC , its diagonal; then in $P-ACD$ and $P-CDE$, | |
| 4. Base $ACD =$ base CDE . | 4. Why? |
| 5. They have the same altitude (the perpendicular from P to $ADEC$). | 5. § 506. |
| 6. Hence $P-ACD = P-CDE$. | 6. § 518. |
| Similarly, in pyramid $C-ABPD$, $C-APB = C-APD$ | |

7. But $C-APB = P-ABC$ and $C-APD = P-ACD$. 7. By identity.
8. Hence $P-ABC = P-ACD = P-CDE$. 8. Ax. 5.
- But these three triangular prisms make up the prism $ABC-DPE$, so $P-ABC = \frac{1}{3}$ of this prism.
9. $\therefore V = \frac{1}{3} bh$. 9. Ax. 6.

Therefore, the volume of a triangular pyramid

~~AAA~~

520. Corollary. *The volume of any pyramid equals one third the product of the base and the altitude.*

By passing planes through the vertex and the diagonals of the base from one vertex of the base, the pyramid is divided into triangular pyramids. The volume of each of these is one third the product of the common altitude by its base. (§ 519.) Hence the sum of these triangular pyramids is one third the altitude times the sum of the bases. (Ax. 1.) But the sum of the bases of the triangular pyramids is the base of the original pyramid.

Ex. 1. Find the volume of a right regular pyramid whose base is an equilateral triangle, each of whose sides is 15 ft., and whose altitude is 20 ft. *645. 3*

Ex. 2. What is the slant height and the lateral area of the pyramid of Exercise 1? *12.91, 296.47*

Ex. 3. The center of a cube is connected with each vertex. How many pyramids are formed? Are they mutually congruent? *8*

Ex. 4. Join any point P within a ~~circle~~ to the eight vertices. Compare the sum of the volumes of one pair of pyramids having bases on opposite sides of the cube with the sum of the volumes of any other such pair.

Ex. 5. Prove theorem § 518 by constructing prisms on each section as upper bases, and continuously increasing the number of parts of h .

Ex. 1. Is any straight line drawn through the midpoint of any diagonal of a parallelepiped and terminating in two opposite faces bisected by the point?

HINT. Pass a plane through AB and CD . Is $\triangle ADM \cong \triangle BCM$?

Ex. 2. Are the four diagonals of a right rectangular parallelepiped equal?

Ex. 3. Is the converse of Exercise 2 true?

Ex. 4. Do the diagonals of any parallelepiped meet in a point?

HINT. Can you pass a plane through any two corresponding diagonals of opposite bases? (Are such diagonals the opposite sides of a parallelogram?)

Ex. 5. AG (Fig. 2) is a cube. A plane is passed through F and the midpoints K and L of AB and BC . What is the ratio of the volume of $F-KBL$ to the volume of the cube?

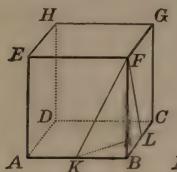


FIG. 2

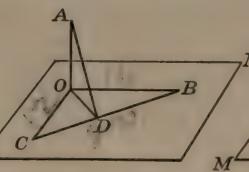


FIG. 3

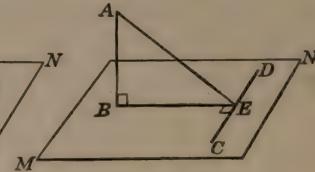


FIG. 4

Ex. 6. In Figure 3, AO is perpendicular to OB and OC , OB is perpendicular to OC , and OD is perpendicular to BC . AO equals $10'$, OB equals $16'$, and OC equals $12'$. Compute the lengths of BC , OD , and AD . (Answers: 20 , 9.6 , and 13.9 .)

Ex. 7. In Figure 4, AB is \perp to MN , and CD , a line in plane MN , is perpendicular to BE . Is AE perpendicular to CD ?

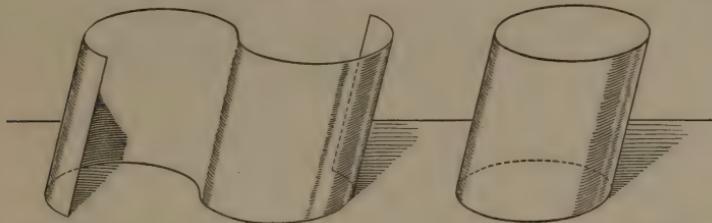
HINT. Assume (or lay off) ED equal to EC . Draw AC , AD , BC , and BD . Is $\triangle ACD$ isosceles?

Ex. 8. The sides of the base of a right regular hexagonal pyramid are each 8 in. The altitude is 9 in. Find the total area and the volume.

432.93 ; area
498.24 vol.

CYLINDERS

521. Cylindrical surface. If a straight line segment moves so that it is always parallel to its original position and so that it constantly touches a fixed curve in a plane not containing the line, it generates a *cylindrical surface*.



If the curve is a closed curve, the cylindrical surface is said to be a *closed cylindrical surface*. The line is the *generatrix* and the curve is the *directrix*. Any one position of the generatrix is an *element*.

522. A cylinder is a solid bounded by a closed cylindrical surface and two parallel planes cutting all the elements. The parallel sections are the *bases*, the *lateral surface* is the curved surface between the bases, and the *altitude* is the distance between the planes of the bases.

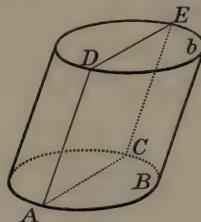
523. Properties of a cylinder. From the preceding definitions, we conclude that :

1. *The elements are parallel to each other.*
2. *The elements are equal*, since parallel lines intercepted by parallel planes are equal.

524. Congruent irregular figures. Any two irregular figures are congruent if every dimension of one can be made to coincide with a corresponding dimension of the other.

Proposition 13. Bases of a Cylinder

525. Theorem. *The bases of a cylinder are congruent.*



Given a cylinder with bases b and B .

To prove that $b \cong B$.

The plan is to show that any and hence every dimension of base b can be made to coincide with a corresponding dimension of base B . (Can you prove it without reading further?)

Proof. Let AC be any dimension of base B . Let AD and CE be elements. Draw DE .

1. AD and CE are $=$ and \parallel .
2. Hence $ACED$ is a \square .
3. Hence $DE = AC$.

That is, any and hence every dimension of base b is equal to and may therefore be made to coincide with a corresponding dimension of base B .

4. $\therefore b \cong B$.

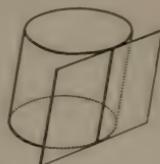
1. § 523.
2. § 157.
3. § 145.

4. § 524.

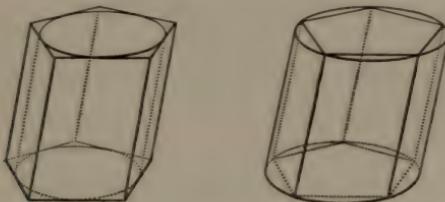
Therefore, the bases of a cylinder are congruent.

526. A **right cylinder** is one whose elements are perpendicular to the bases; an **oblique cylinder** is one whose elements are not perpendicular to the bases; a **circular cylinder** is one whose bases are circles. A **right circular cylinder** is sometimes called a *cylinder of revolution*, since it may be generated by a rectangle revolving about one of its sides as an axis.

527. A plane is tangent to a cylinder if it passes through an element and no other point of the cylinder.



528. Circumscribed prisms. If all the lateral faces of a prism are tangent to a cylinder and if the bases lie in the same planes as the bases of the cylinder, the prism is *circumscribed* about the cylinder, and the cylinder is *inscribed* in the prism.



529. Inscribed prism. A prism is an inscribed prism if all its lateral edges are elements of the cylinder and its bases lie in the planes of the bases of the cylinder. In that case the cylinder is circumscribed about the prism.

530. Prism approaches cylinder. If a prism of equilateral base is inscribed in a cylinder or circumscribed about a cylinder and the number of its faces is continuously increased, it is evident that the lateral surface, the volume, and the bases of the prism approach as limits the lateral surface, the volume, and the bases respectively of the cylinder.

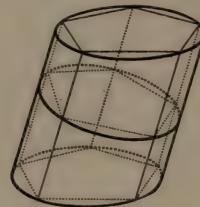
Ex. 1. If any two elements of a cylinder have the end points in each base connected, what kind of figure is formed?

Ex. 2. If a plane is tangent to a cylinder, is it perpendicular to the bases?

Ex. 3. If a prism is inscribed in a circular cylinder, do the vertices of the bases of the prism lie in the circumferences of the bases of the cylinder?

Proposition 14. Area of a Cylinder

531. Theorem. *The lateral area of a cylinder is equal to the product of an element and the perimeter of a right section.*



Given a cylinder with lateral area A , element l , and perimeter of right section p .

To prove that $A = lp$.

The plan is to inscribe a prism, continuously increase the number of sides, and apply the theory of limits.

Proof. Inscribe any prism of equilateral bases in the cylinder. Let S represent its lateral area and let p' be the perimeter of its right section.

1. Then $S = lp'$.

Now if the number of sides is continuously increased, S will approach A as its limit and p' will approach p as its limit. That is,

2. $S \rightarrow A$,

3. and $p' \rightarrow p$.

4. Hence $lp' \rightarrow lp$.

Now the variables S and lp' are always equal; therefore their limits are equal; that is,

5. $A = lp$.

1. § 490.

2. § 530.

3. § 530.

4. § 517.

5. § 387. 517

Therefore, the lateral area of a cylinder

532. Corollary. *The lateral area of a cylinder of revolution equals $2\pi rh$, in which r equals the radius of the cylinder and h equals the element. (The perimeter of a circle is $2\pi r$.)*

Proposition 15. Volume of a Cylinder

533. Theorem. *The volume of a cylinder is the product of the base and the altitude.*



Given a cylinder with volume V , area of the base b , and altitude h .

To prove that $V = bh$.

The plan is to inscribe a prism in the cylinder and apply the theory of limits.

Proof. Inscribe a prism of equilateral base in the cylinder. Let V' represent its volume and b' the area of its base.

1. Then $V' = b'h$.

Now increase continuously the number of faces of the prism.

2. Then $V' \rightarrow V$,
3. and $b' \rightarrow b$.
4. Hence $b'h \rightarrow bh$.

Now, since variables V' and $b'h$ are always equal, their limits are equal; that is,

5. $V = bh$.

1. § 505.

2. § 530.

3. § 530.

4. § 387, 2. 517

5. § 387, 1. 517

Therefore, the volume of a cylinder

534. Corollary. *In a cylinder of revolution, $V = \pi r^2 h$.*

Ex. 1. Solve $V = \pi r^2 h$ for r ; for h . - $R = \sqrt{\frac{V}{\pi}}$

Ex. 2. Find V in the above formula when $r = 4$ in. and $h = 10$ in.

502.6

EXERCISES

Ex. 1. Is the section of a cylinder containing an element a rectangle? a parallelogram?

Ex. 2. Is the right section of a circular cylinder a circle?

Ex. 3. Is every section of a cylinder of revolution a circle?

Ex. 4. Is a right section of a cylinder of revolution a circle?

Ex. 5. Is the section of a cylinder made by a plane parallel to the base congruent to the base?

Ex. 6. What is the locus of points in space at a given distance, d , from a straight line, l ?

Ex. 7. What is the locus of points in space at a given distance, d , from a given line, l , and at a given distance d' , from a plane, p , containing this line? Under what conditions is there no locus?

Ex. 8. What is the ratio of the volume of a regular triangular prism to the volume of the circumscribed cylinder?

Ex. 9. What is the ratio of the volume of a square prism to the volume of the circumscribed cylinder?

Ex. 10. If the number of faces of the prism is indefinitely increased, what does the ratio of its volume to the volume of the circumscribed cylinder approach as its limit?

Ex. 11. Find the capacity in gallons of a tank car 32 ft. long and 5 ft. in diameter. (Use 7.48 gal. per cu. ft.)

Ex. 12. Would the straight line joining the centers of the bases of a circular cylinder pass through the center of the section made by a plane parallel to the bases?

Ex. 13. Prove that the total area, T , of a cylinder is equal to $2\pi rh + 2\pi r^2$.

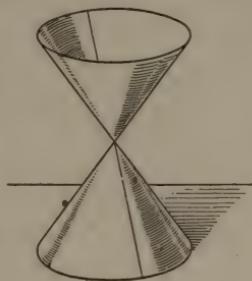
Ex. 14. Remove the largest monomial factor from the formula of Exercise 13 and state the resulting formula in words.

Ex. 15. A 50-gallon gasoline tank is 26" in diameter. What is its height?

Ex. 16. Is the altitude of a cylinder equal to its lateral edge?

CONES

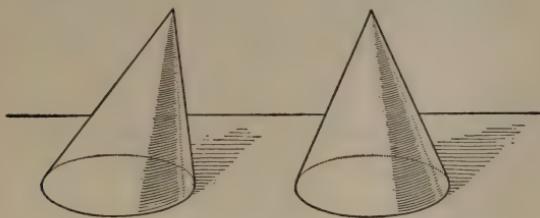
535. Conical surface. If a straight line moves so that it always contains a fixed point and always touches a fixed curve not containing the point, it generates a *conical surface*. The straight line is the *generatrix*, the fixed curve is the *directrix*, and the fixed point is the *vertex*. Each position of the straight line is an *element* of the conical surface.



It is evident that, since the straight line is indefinite in length, it generates two symmetric parts, one on each side of a plane through the vertex which does not otherwise cut the surface. These two parts are called the *näppes*.

536. A cone is a solid bounded by one nappe of a conical surface and by a plane cutting all its elements. The plane section of the conical surface is the *base* of the cone, and the perpendicular distance from the vertex to the base is the *altitude* of the cone.

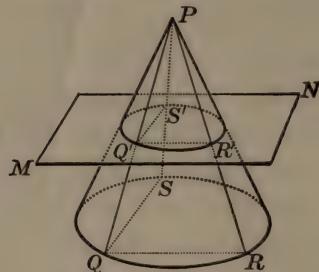
537. A circular cone is one whose base is a circle. The *axis* of a circular cone is the line joining its vertex to the center of its base.



538. A right circular cone, or a cone of revolution, is a circular cone whose axis is perpendicular to its base. Any element of a cone of revolution is called its *slant height*.

Proposition 16. Section of a Cone

539. Theorem. *In any cone a section made by a plane parallel to the base is similar to the base.*



Given a cone with base b and section s formed by plane $MN \parallel$ to the base.

To prove that $s \sim b$.

The plan is to show that any two dimensions of b are matched by two corresponding dimensions of s and that the ratios of the corresponding dimensions are equal.

Proof. Let PQ , PR , and PS be any three elements of the cone, cutting MN at Q' , R' , and S' respectively. Draw QR , QS , $Q'R'$, and $Q'S'$.

- | | |
|--|--|
| <ol style="list-style-type: none"> 1. $Q'R'$ is \parallel to QR and $Q'S'$ is \parallel to QS. 2. Hence $\triangle PQ'R' \sim \triangle PQR$, and $\triangle PQ'S' \sim \triangle PQS$. 3. Hence $\frac{Q'R'}{QR} = \frac{PQ'}{PQ} = \frac{Q'S'}{QS}$. 4. Since any two dimensions in b are matched by corresponding dimensions in s, and the ratios of the two corresponding dimensions are equal, the section is similar to the base. | <ol style="list-style-type: none"> 1. § 444. 2. § 310. 3. § 304. 4. § 349. |
|--|--|

Therefore, in any cone

540. Conic sections. A section formed by the intersection of a plane and a conic surface of revolution is called a *conic section*. As shown by Proposition 16, the conic sec-



FIG. 1



FIG. 2



FIG. 3



FIG. 4

tion in Figure 1 is a *circle*. That shown in Figure 2 is an *ellipse*. When the plane is parallel to the axis, the section is a *hyperbola* (Fig. 3). When the plane is parallel to an element, the section is a *parabola*, which is exemplified by the path of a projectile in a vacuum (Fig. 4).

541. Plane tangent to cone. A plane that contains one element of a cone and no other point on the cone is tangent to the cone. It is tangent to the base also.

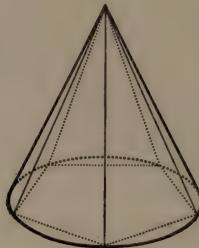
542. Circumscribed and inscribed pyramids. A pyramid is *circumscribed* about a cone if all its lateral faces are tangent to the cone and its base lies in the plane of the base of the cone.

A pyramid is *inscribed* in a cone if all its lateral edges lie in the lateral surface of the cone and its base lies in the plane of the base of the cone.

543. Pyramid approaches cone. If a pyramid of equilateral base is circumscribed about or inscribed in a cone and has the number of its faces continuously increased, the volume, the lateral area, the perimeter and area of the base, and the slant height of the pyramid are variables which approach as limits the volume, the lateral area, the perimeter and area of the base, and the slant height of the cone.

Proposition 17. Area of a Cone

544. Theorem. *The lateral area of a cone of revolution is equal to half the product of the slant height and the circumference of its base.*



Given a cone of revolution with lateral area A , circumference of the base c , and slant height l .

To prove that $A = \frac{1}{2} cl$.

The plan is to inscribe a regular pyramid in the cone and apply the theory of limits. (Can you prove it?)

Proof. Inscribe a regular pyramid in the cone, and let A' be its lateral area, l' its slant height, and p the perimeter of its base.

$$1. A' = \frac{1}{2} l' p.$$

Now if the number of faces is continuously increased,

$$2. A' \rightarrow A, l' \rightarrow l, \text{ and } p \rightarrow c.$$

$$3. \text{ Hence } \frac{1}{2} l' p \rightarrow \frac{1}{2} cl.$$

Since the variables A' and $\frac{1}{2} l' p$ are always equal, their limits are equal; that is,

$$4. A = \frac{1}{2} cl.$$

1. § 512.

2. § 543.

3. § 517, 2.

4. § 517, 1.

Therefore, the lateral area of a cone of revolution . . .

545. Corollary. *The total area of a cone of revolution is $\pi r l + \pi r^2 = \pi r(l + r)$. (§ 390, Ax. 1.)*

546. Corollary. *The lateral areas of two similar cones of revolution are to each other as the squares of any two corresponding lines.*

$$A = \pi r l \text{ and } A' = \pi r' l'. \quad (\S\ 544.) \quad \therefore \frac{A}{A'} = \frac{\pi r l}{\pi r' l'} = \frac{r l}{r' l'}.$$

But $\frac{r}{r'} = \frac{l}{l'} = \frac{h}{h'}$. Hence $\frac{A}{A'} = \frac{r^2}{r'^2} = \frac{l^2}{l'^2} = \frac{h^2}{h'^2}$. (Ax. 6.)

547. The **frustum of a cone** is the portion of the cone included between the base and a section made by a plane parallel to the base. The section is the upper base, and the base of the cone is the lower base of the frustum.

The *altitude* of the frustum is the perpendicular distance between the bases.

The *slant height* of the frustum of a cone of revolution is that portion of an element included between the bases.

If the frustum of an equilateral pyramid is circumscribed about or inscribed in the frustum of a cone, and the number of its faces is continuously increased, the volume, the lateral area, the perimeters and areas of the bases, and the slant height of the frustum are variables that approach as their limits the volume, the lateral area, the perimeters and areas of the bases, and the slant height of the frustum of the cone.

Ex. 1. Is the section of a cone made by a plane through the vertex a triangle?

Ex. 2. Is the section of a circular cone made by a plane cutting all the elements a circle?

Ex. 3. Does the altitude of a right circular cone meet the base at the center?

Ex. 4. If a right triangle is revolved about one of its legs as an axis, is the solid generated a cone? If so, what kind?

Ex. 5. In what respect are a right circular cylinder and a right circular cone alike?

Proposition 18. Area of Frustum of a Cone

548. Theorem. *The lateral area of a frustum of a cone of revolution is equal to half the sum of the circumferences of the bases multiplied by the slant height of the frustum.*



Given a frustum of a cone of revolution with lateral area A , circumferences c and C of the upper and lower base respectively, and slant height l .

To prove that $A = \frac{1}{2}l(c + C)$.

Proof. Inscribe the frustum of a regular pyramid in the given frustum. Let p and P be the perimeters of its upper and lower base respectively, l' its slant height, and A' its lateral area.

1. Then $A' = \frac{1}{2}l'(p + P)$.

If the number of faces of the frustum of the pyramid is continuously increased,

2. $A' \rightarrow A$, $l' \rightarrow l$, $p \rightarrow c$, and $P \rightarrow C$.

3. Hence $\frac{1}{2}l'(p + P) \rightarrow \frac{1}{2}l(c + C)$.

Now, since the variables A' and $\frac{1}{2}l'(p + P)$ are always equal, their limits are equal,

4. $\therefore A = \frac{1}{2}l(c + C)$.

1. § 516.

2. § 547.

3. § 517, 2.

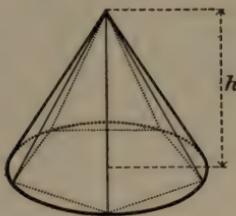
4. § 517, 1.

Therefore, the lateral area of a frustum of a cone . . .

549. Corollary. *The lateral area of the frustum of a cone of revolution is equal to the circumference generated by the midpoint of an element multiplied by the slant height of the frustum.*

Proposition 19. Volume of a Cone

550. Theorem. *The volume of a cone is equal to one third the product of the base and the altitude.*



Given a cone with volume V , altitude h , and area b of the base.

To prove that $V = \frac{1}{3}bh$.

The plan is to inscribe a pyramid in the cone and apply the theory of limits. (Can you prove it without reading further?)

Proof. Let P be a pyramid of equilateral base inscribed in the cone. Let V' be its volume and b' the area of its base.

$$1. V' = \frac{1}{3} b'h.$$

Now let the number of faces of the pyramid be continuously increased.

$$2. \text{ Then } V' \longrightarrow V,$$

$$3. \text{ and } b' \longrightarrow b.$$

$$4. \text{ Hence } \frac{1}{3} b'h \longrightarrow \frac{1}{3} bh.$$

$$5. \therefore V = \frac{1}{3} bh.$$

1. Why?

2. Why?

3. Why?

4. Why?

5. § 517, 1.

Therefore, the volume of a cone

Ex. 1. If a right triangle is revolved about its hypotenuse as an axis, what kind of a surface is generated?

Ex. 2. Through what point in the altitude of a cone must a plane parallel to the base be passed to divide the cone into two equivalent volumes?

EXERCISES

Ex. 1. Prove that the volume of a frustum of a pyramid is expressed correctly by the formula $V = \frac{1}{3} h(B + b + \sqrt{Bb})$.

HINT. Express its volumes as the difference between the volumes of the whole pyramid and the part cut off by the upper base of the frustum. Let $V, V', V'',$ and h, h', h'' , be the volumes and the altitudes of the frustum, pyramid, and part removed, respectively. Let B and b be the lower base and the upper base of the frustum. Then $h'' = h' - h$. Now $V = V' - V'' = \frac{1}{3}[Bh' - b(h' - h)]$.

$$V = \frac{1}{3}[h'(B - b) + bh], \text{ but } \frac{B}{b} = \frac{h'^2}{(h' - h)^2} \text{ or } \frac{\sqrt{B}}{\sqrt{b}} = \frac{h'}{h' - h}. \text{ Solve}$$

for h' and substitute in the equation for V above. Remember that $B - b = (\sqrt{B} + \sqrt{b})(\sqrt{B} - \sqrt{b})$.

Ex. 2. Can you prove that the volume of a frustum of a cone of revolution is expressed correctly by the formula $V = \frac{1}{3} \pi h(R^2 + r^2 + Rr)$?

HINT. Inscribe the frustum of a regular pyramid in the frustum of the cone and apply the theory of limits.

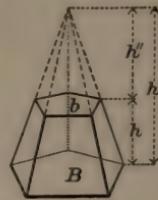
Ex. 3. A cone and a cylinder have the same base and volume. How do their altitudes compare?

Ex. 4. Express the altitude of a right circular cone in terms of the radius of its base and its slant height.

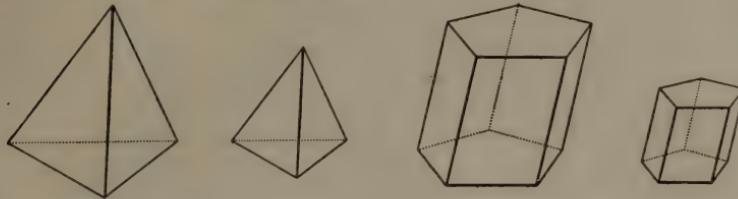
Ex. 5. Express the distance from the midpoint of the upper base of a circular cylinder of revolution to any point in the circumference of the lower base in terms of h and r .

Ex. 6.¹ If a circular cone is cut by a plane parallel to the base, (1) all the elements and the altitude of the given cone are divided proportionally, (2) the radii of the bases of the two cones are to each other as the altitudes of the two cones, and (3) the areas of the bases of the two cones are to each other as the squares of the altitudes of the two cones.

¹ This exercise often appears as a proposition.



551. Similar polyhedrons. Two convex¹ polyhedrons are similar if they have the same number of faces similar each to each and similarly placed, and if their corresponding polyhedral angles are equal.



Ex. 1. State under what conditions the two pairs of polyhedrons shown here are similar.

Ex. 2. If you double all the dimensions of a right rectangular parallelepiped, by what ratio have you increased the total area? the volume?

Ex. 3. Can you prove that the shortest line in the planes of the walls of your room from an upper corner to the diagonally opposite lower corner is equal to $\sqrt{h^2 + (w + l)^2}$?

HINT. Take a sheet of paper and fold it to represent a side and an end of your room. Then straighten out the sheet of paper, draw the line, note the dimensions, and compute the length.

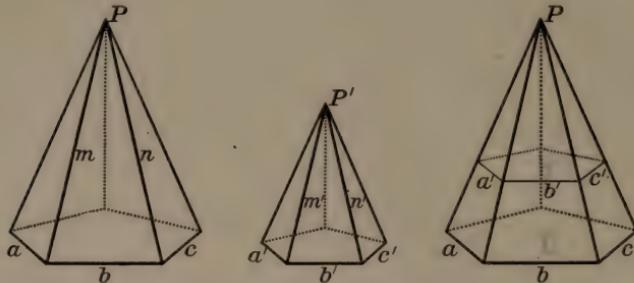
552. Properties of similar polyhedrons. From the definition of similar convex polyhedrons, it follows that the corresponding face angles, dihedral angles, and polyhedral angles of similar polyhedrons are equal and that their corresponding edges are proportional. Two similar polyhedrons can be divided into the same number of tetrahedrons, similar each to each, and similarly placed.

Ex. 4. Two solids are similar when they have the same _____. They are _____ when they have the same size. They are congruent when they have the same _____ and _____.

¹A convex polyhedron is one all of whose polyhedral angles are convex.

Proposition 20. Volumes of Similar Pyramids

553. Theorem. *The volumes of any two similar pyramids are to each other as the cubes of any two corresponding edges or of the two altitudes.*



Given any two similar pyramids P and P' , with edges a, b, \dots, m, n, \dots , and $a', b', \dots, m', n', \dots$; volumes V and V' ; bases B and B' ; and altitudes h and h' .

To prove that $V/V' = a^3/a'^3 = m^3/m'^3 = h^3/h'^3$.

Proof. Place P' in P so that polyhedral $\angle P'$ coincides with its equal polyhedral $\angle P$.

- | | |
|---|----------------------|
| 1. a is \parallel to a' , b is \parallel to b' , etc. | 1. § 298. |
| 2. Hence B is \parallel to B' . | 2. § 446. |
| 3. Hence $a/a' = b/b'$, etc., $= m/m' = n/n'$, etc., $= h/h'$. | 3. § 510. |
| 4. Now $V/V' = Bh/B'h'$. | 4. § 520, Ax. 4. |
| 5. But $B/B' = a^2/a'^2$, | 5. § 339. |
| 6. and $h/h' = a/a'$. | 6. See No. 3. |
| 7. $\therefore V/V' = a^3/a'^3 = m^3/m'^3 = h^3/h'^3$. | 7. Ax. 6 and Ax. 15. |

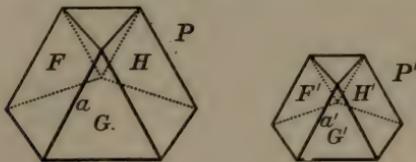
Therefore, the volumes of any two similar pyramids . . .

554. Corollary. *The volumes of two similar cones are to each other as the cube of any two corresponding lines.*

Inscribe a regular pyramid in the cone, continuously increase the number of faces, and apply the theory of limits.

Proposition 21. Similar Polyhedrons

555. Theorem. *The areas of two similar polyhedrons are to each other as the squares of any two corresponding edges.*



Given any two similar polyhedrons P and P' , with areas A and A' ; faces F, G, H , etc., and F', G', H' , etc., and any two corresponding edges a and a' , as shown.

To prove that $\frac{A}{A'} = \frac{a^2}{a'^2}$.

The plan is to show that $\frac{A}{A'} = \frac{F}{F'} = \frac{a^2}{a'^2}$.

Proof.

- | | |
|---|--------------|
| 1. The ratio of any two corresponding edges = $\frac{a}{a'}$. | 1. § 552. |
| 2. Hence $\frac{F}{F'} = \frac{a^2}{a'^2}$, $\frac{G}{G'} = \frac{a^2}{a'^2}$, etc. | 2. § 339. |
| 3. Hence $\frac{F}{F'} = \frac{G}{G'} = \frac{H}{H'}$, etc. = $\frac{a^2}{a'^2}$. | 3. Ax. 5. |
| 4. Hence $\frac{F+G+H, \text{ etc.}}{F'+G'+H', \text{ etc.}} = \frac{a^2}{a'^2}$. | 4. § 287, 8. |
| 5. That is, $\frac{A}{A'} = \frac{a^2}{a'^2}$. | 5. Ax. 6. |

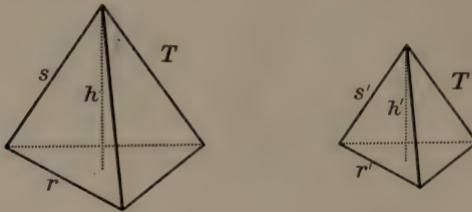
Therefore, the areas of two similar polyhedrons

556. Corollary. *The areas of similar cylinders are to each other as the square of any two corresponding lines.*

Inscribe regular prisms with the same number of faces in the cylinders. These will be similar polyhedrons. Increase the number of faces continuously and apply the theory of limits.

Proposition 22. Volumes of Similar Tetrahedrons

557. Theorem. *The volumes of any two similar tetrahedrons are to each other as the cubes of any two corresponding sides.*



Given any two tetrahedrons T and T' , with altitudes h and h' , any two pairs of corresponding sides r and r' , and s and s' , and bases B and B' .

To prove that $\frac{T}{T'} = \frac{s^3}{s'^3}$.

The plan is to show that $\frac{T}{T'} = \frac{\frac{1}{3}Bh}{\frac{1}{3}B'h'} = \frac{h^3}{h'^3} = \frac{s^3}{s'^3}$.

Proof.

$$1. \quad V = \frac{1}{3}Bh \text{ and } V' = \frac{1}{3}B'h'.$$

$$2. \quad \text{Hence } \frac{V}{V'} = \frac{Bh}{B'h'}.$$

$$3. \quad \text{But } \frac{B}{B'} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2},$$

$$4. \quad \text{and } \frac{h}{h'} = \frac{s}{s'}.$$

$$5. \quad \text{Hence } \frac{V}{V'} = \frac{s^3}{s'^3}.$$

1. § 519.

2. Ax. 4.

3. § 334.

4. Why?

5. Why?

Ex. 1. If two tetrahedrons are similar, are they regular? If regular, are they similar?

Ex. 2. Can vertical trihedral angles be congruent?

558. Corollary. *The volumes of any two similar polyhedrons are to each other as the cubes of any two corresponding sides.*

Assume the polyhedrons to be divided into tetrahedrons, similar each to each, and use § 287, 8.

559. Corollary. *The volumes of two similar cylinders are to each other as the cubes of any two corresponding lines.*

Inscribe equilateral prisms of the same number of sides in the cylinder, continuously increase the number of faces, and apply the theory of limits.

NUMERICAL EXERCISES

Ex. 1. Find the volume, the total surface, and the length of a diagonal of the right rectangular parallelepiped whose dimensions are 10 ft., 8 ft., and 6 ft.

Ex. 2. Find the surface and the diagonal of a cube whose volume is 729 cu. in.

Ex. 3. A freight car whose inside dimensions are: length 33 ft., width 8 ft. 4 in., is filled with wheat to the depth of 4 ft. 6 in. How many bushels does it contain? (Use $\frac{5}{4}$ cu. ft. per bushel.)

Ex. 4. Find the total area and the volume of a regular triangular prism, the edge of whose base is 8 in. and altitude 10 in.

Ex. 5. Find the area and the volume of a right regular quadrangular pyramid, the edge of whose base is 12 ft. and altitude 10 ft.

Ex. 6. Find the volume of a triangular prism whose altitude is 16 ft. and whose base is a triangle whose sides are 12 ft., 15 ft., and 10 ft.

Ex. 7. Find the total area and the volume of a regular tetrahedron whose edge is 10 in.

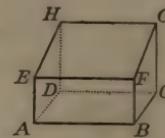
Ex. 8. Find the area and the volume of a right regular hexagonal pyramid whose altitude is 20 ft. and the edge of whose base is 12 ft.

EXERCISES

Ex. 1. How many cubic yards of material is removed in excavating a basement 34 ft. by 26 ft. and 8 ft. deep?

Ex. 2. At 35 cu. ft. a ton, how long must a coal bin be that is 10 ft. wide and 6 ft. deep to hold 20 tons?

Ex. 3. A truncated prism has a square base 20 in. on each side. The lateral edges are perpendicular to the base. $EA = FB = 8$ in., and $HD = GC = 14$ in. Find the total area, volume, and diagonals.



Ex. 4. The edges of a right rectangular parallelepiped meeting at a vertex are 8 in., 10 in., and 12 in. Find the total area and the volume.

Ex. 5. Find the area and the volume of a right circular cone whose slant height is 12 in. and whose radius of base is 8 in.

Ex. 6. Find the area and the volume of a right circular cone whose altitude is 10 in. and whose radius of base is 12 in.

Ex. 7. How many gallons of water ($7\frac{1}{2}$ to the cubic foot) will a right cylindrical hot-water tank hold, whose altitude is 4 ft. 8 in. and whose inside diameter is 15 in.?

Ex. 8. How many 4 in. water mains will a 36 in. trunk main supply?

Ex. 9. At 437 lb. per cubic foot, what is the weight of an iron shaft 4 in. in diameter and 12 ft. long?

Ex. 10. Find in terms of a the volume of a triangular pyramid, all of whose edges are a .

Ex. 11. How many cubic yards of earth must be removed in digging a well 5 ft. in diameter and 45 ft. in depth?

Ex. 12. A quadrangular pyramid with a square base whose side is 20 ft. has for its faces equilateral triangles. What is its altitude? slant height? lateral area? volume?

Ex. 13. Make a drawing of and describe accurately the locus of a point 4 in. from a given line and 8 in. from a point on that line.

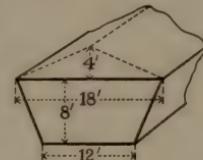
Ex. 14. A Winchester bushel is defined as the contents of a cylindrical vessel $18\frac{1}{2}$ in. in diameter and 8 in. deep. How many cubic inches is that?

Ex. 15. How many feet of copper wire one-tenth inch in diameter will a cubic foot of copper make?

Ex. 16. What is the volume of a cylindrical silo 12 ft. in diameter and 20 ft. high? At 48 cu. ft. a ton, how many tons of ensilage will it hold?

Ex. 17. Ice is .92 as heavy as water. What is the weight of a block of ice 20 in. by 18 in. by 12 in.? (Water weighs 62.5 lb. per cubic foot.)

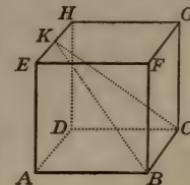
Ex. 18. A corn crib 120 ft. long is 8 ft. high, 12 ft. wide at the bottom, 18 ft. at the top, and is piled up to the height of 4 ft. above the top of the crib. Allowing 1 bu. for each $1\frac{1}{2}$ cu. ft., in this case, how many bushels does it contain?



Ex. 19. The altitude of a right square pyramid is 10 ft., the side of the base is 8 ft. Find the volume of the inscribed cone.

Ex. 20. AG is a cube whose edge is 16 in. K is the midpoint of EH . Find the area of $\triangle BKC$.

Ex. 21. How many ice-cream cones 2 inches in diameter and 2 inches high can be filled from a gallon of ice cream? (7.5 gal. = 1 cu. ft.)



Ex. 22. Given the volume and the altitude of a right circular cone, express the slant height in terms of these. Express the total area.

Ex. 23. A room is heated by five steam pipes each 32 ft. long and 3 in. in diameter. What is the area in square feet of the radiating surface? (Neglect the connections at the ends.)

Ex. 24. Given the volume and the radius of the base of a right circular cylinder. Find in terms of these the altitude and the lateral area.

EXERCISES

Ex. 1. A railroad tunnel has for its cross-section a rectangle 16 ft. by 10 ft. surmounted by a semicircle 16 ft. in diameter. The tunnel is 520 ft. long. How many cubic yards of earth were removed?

Ex. 2. If a regular hexagonal prism is inscribed in a cylinder of revolution whose radius is 8 in. and whose altitude is 12 in., what part of the volume of the cylinder is the volume of the prism?

Ex. 3. If a regular hexagonal prism is inscribed in a cylinder of revolution whose radius is r and whose altitude is h , what part of the volume of the cylinder is the volume of the prism?

Ex. 4. Why are the results of Exercises 2 and 3 the same?

Ex. 5. Find the volume of the frustum of a right square pyramid, the sides of whose bases are 18 ft. and 12 ft. and whose altitude is 20 ft.

Ex. 6. Find the number of cubic feet in a log 32 ft. long and 40 in. in diameter at the small end and 54 in. at the large end.

Ex. 7. Is the volume of an oblique prism equal to the volume of a right prism whose base is a right section of the oblique prism and whose altitude is equal to the altitude of the oblique prism?

Ex. 8. If two prisms have the same base and equal altitudes, do they have the same volume? the same area?

Ex. 9. If you multiply each of the dimensions of a parallelepiped by two, do you multiply the area by two? the volume?

Ex. 10. A brick smokestack 130 ft. high is 20 ft. in diameter at the ground and 12 ft. at the top. It has a cylindrical flue 4 ft. in diameter throughout its length. Allowing 22 bricks per cubic foot, how many bricks are there in it?

Ex. 11. A grindstone is 3 ft. 2 in. in diameter. It is 3 in. thick. It has a square hole in the center 3 in. on the side. At 142 lb. per cubic foot, what is its weight?

REVIEW

Ex. 1. How many edges (e) has a tetrahedron? How many vertices (v)? How many faces (f)?

Ex. 2. How many edges (e) has a triangular prism? How many vertices (v)? How many faces (f)?

Ex. 3. How many edges (e) has a cube? How many vertices (v)? How many faces (f)?

Ex. 4. Will the formula $e = v + f - 2$ hold in Exercise 1? Exercise 2? Exercise 3?

Ex. 5. Try the formula of Exercise 4 for other polyhedrons.

Ex. 6. How does the square of the diagonal of a cube compare with the total area?

Ex. 7. A cylinder of revolution is inscribed in a regular hexagonal prism. What is the ratio of the volume of the cylinder to that of the prism?

Ex. 8. What are the dimensions of a quart tomato can if its altitude is equal to the diameter of its base? (1 qt. = 57.75 cu. in.)

Ex. 9. A piece of tin in the form of a sector of a circle of radius 10 in. is rolled into a cone. If the central angle of the sector is 210° , what is the volume of the resulting cone?

Ex. 10. Prove that if a quadrilateral has each of its vertices out of the plane of the remaining three, the lines joining the midpoint of the sides of the quadrilateral form a parallelogram.

Ex. 11. Find the lateral area of the frustum of a cone of revolution with the following dimensions: $R = 18$ in., $r = 10$ in., $h = 12$ in.

Ex. 12. Find the volume of the above frustum, given $V = \frac{1}{3}\pi h (R^2 + r^2 + Rr)$.

Ex. 13. If lines are drawn from the vertex of a tetrahedron to the midpoints of the sides of the base and planes passed containing each two of these lines, what part of the volume of the tetrahedron is the pyramid formed by the planes and the base?

BOOK EIGHT

SPHERES

560. A **sphere** is a curved surface¹ all points of which are equidistant from a point within called the *center*. A sphere may be generated by the revolution of a semicircle about its *diameter*.

The diameter of a sphere is the line joining two points of the sphere and passing through the center. The *radius* is the distance from the center to the surface.

561. A **great circle** of a sphere is one that has for its diameter a diameter of the sphere. The plane of a great circle passes through the center of the sphere. All great circles of a sphere are equal. A *small circle* is one that has for its diameter a chord of the sphere less than the diameter.

562. The *distance* between two points on a sphere is the minor arc of a great circle through them. Equal minor arcs (or major arcs) of great circles of the same or equal spheres can, of course, be made to coincide, and conversely.

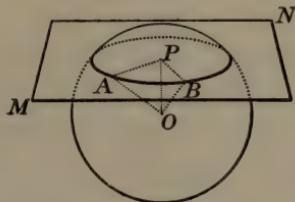
563. Obviously any number of great circles can pass through two points if they are at the ends of a diameter. However, only one minor arc of a great circle can pass through two given points that are not at the ends of a diameter, and, of course, two minor arcs of great circles cannot intersect in more than one point.

564. Tangent lines and planes. A *line is tangent* to a sphere, if it has but one point in common with the sphere. A *plane is tangent* to a sphere, if it has but one point in common with the sphere.

¹ A sphere is sometimes thought of as the space enclosed by the curved surface.

Proposition 1. Section of a Sphere

565. Theorem. *If a plane intersects a sphere, the intersection is a circle.*



Given the sphere whose center is at O , and a plane MN intersecting the sphere.

To prove that the intersection is a circle.

The plan is to prove that the foot of the \perp to the plane from O is equidistant from any two points of the intersection.

Proof. From the center O draw $OP \perp$ to MN , meeting MN at P . Draw PA and PB from P to any two points on the intersection and draw AO and BO .

- | | |
|---|-------------------------------|
| 1. Rt. $\triangle OPA \cong$ rt. $\triangle OPB$. | 1. Why?
2. Why?
3. Why? |
| 2. Hence $PA = PB$. That is, any two points on the intersection are equidistant from P . | |
| 3. \therefore the intersection is a circle. | |

Therefore, if a plane intersects a sphere, the intersection is a circle.

566. Corollary. *A perpendicular from the center of a sphere to any intersecting plane meets the circle of intersection at its center.*

567. Corollary. *Through any two points on a sphere that are not the end points of a diameter one and only one minor arc of a great circle can be drawn.*

These two points and the center of the sphere determine a plane which intersects the sphere in a circle.

568. Corollary. *Through any three points on a sphere one and only one circle can be drawn.*

These three points determine a plane and determine a circle on the plane.

569. Poles of a circle. The end points of the diameter perpendicular to the plane of a circle of a sphere are called the *poles* of the circle; this diameter is the *axis* of the circle.

570. The *spherical distance* between two points on a sphere is the length of the minor arc of a great circle joining them.

571. The spherical distance from any point on a circle of a sphere to its nearest pole is the *polar distance* of the circle.

572. A quadrant is one fourth of a great circle.

Ex. 1. Are diameters of equal spheres equal?

Ex. 2. Can two or more great circles of a sphere have the same line segment as a diameter?

Ex. 3. Can two minor circles of a sphere have the same line segment as a diameter?

Ex. 4. Can a straight line cut a sphere in more than two points?

Ex. 5. If a radius of a sphere is perpendicular to a line, is the line tangent to the sphere?

Ex. 6. Does a circle of a sphere cut it into two equal parts? Does a great circle of a sphere?

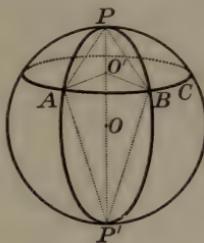
Ex. 7. Under what conditions will the intersections of two parallel planes with a sphere be equal circles?

Ex. 8. Can more than one line be tangent to a sphere at a given point?

Ex. 9. May more than one sphere be passed through two given points? through three? through four?

Proposition 2. Polar Distance

573. Theorem. *The spherical distances of all points on a circle of a sphere from either pole of the circle are equal.*



Given a sphere O with circle C having poles P and P' , and A and B , any two points on the circle, and PAP' and PBP' arcs of great circles from the poles P , P' , through A and B respectively.

To prove that arc $PA = \text{arc } PB$ and that arc $P'A = \text{arc } P'B$.

The plan is to prove that the chords of great circles subtending the arcs are equal. (Can you prove it?)

Proof. Draw AO' and BO' radii of the circle; draw PP' , AP , BP , AP' , and BP' .

- | | |
|--|--|
| 1. $AO' = BO'$.
2. $PO'P'$ is \perp to the plane of the circle.
3. Hence $PA = PB$ and $P'A = P'B$.
4. \therefore arc $PA = \text{arc } PB$ and arc $P'A = \text{arc } P'B$. | 1. Why?
2. § 569.
3. § 430.
4. § 188. |
|--|--|

Therefore, the spherical distances of all points

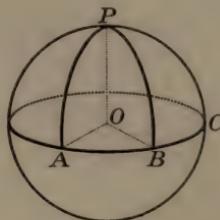
574. Corollary. *The polar distance of a great circle is a quadrant.*

Ex. 1. What is the polar distance of the 49th parallel north latitude?

Ex. 2. What parallels of latitude are each equal to half the equator?

Proposition 3. Poles of a Great Circle

575. Theorem. *On a sphere, a point which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is a pole of the great circle passing through these points.*



Given point P a quadrant's distance from A and from B , two points on a sphere O , not at the extremities of a diameter, and great circle C through A and B .

To prove that P is the pole of great circle C .

The plan is to prove that PO is \perp to the plane of circle C by proving that PO is \perp to two intersecting radii OA and OB .

Proof. Draw PO , OA , and OB .

- | | |
|---|-----------------------------------|
| 1. PO is \perp to AO and to OB .
2. $\therefore PO$ is \perp to plane AOB .
3. Hence PO is the pole of great circle C . | 1. § 237.
2. Why?
3. § 569. |
|---|-----------------------------------|

Therefore, on a sphere,

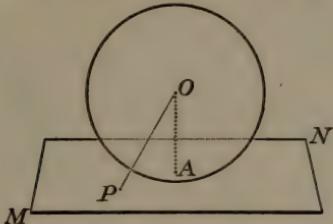
576. A sphere is *inscribed in a polyhedron*, if it is tangent to every face of the polyhedron. The polyhedron in that case is *circumscribed about the sphere*.

577. A sphere is *circumscribed about a polyhedron*, if all the vertices of the polyhedron lie in the surface of the sphere. In that case the polyhedron is *inscribed in the sphere*.

Ex. 1. If the radius of a sphere is 20 inches, what is the length of a circle 15 inches from the center?

Proposition 4. Plane Tangent to a Sphere

578. Theorem. *A plane perpendicular to the radius of a sphere at its outer extremity is tangent to the sphere.*



Given sphere O , with the plane $MN \perp$ to radius OA at A .
To prove that MN is tangent to the sphere O .

The plan is to prove that P , any point in MN other than A , lies outside the sphere. (Can you prove it?)

Proof. Draw OP from O to any point P in MN other than A .

- | | |
|--|--|
| 1. OA is \perp to MN .
2. Hence OP is longer than OA .
3. Hence P lies without the sphere, that is, no
point other than A lies in the sphere.
4. $\therefore MN$ is tangent to the sphere. | 1. Given.
2. § 432.
3. § 560.
4. § 564. |
|--|--|

Therefore, a plane perpendicular

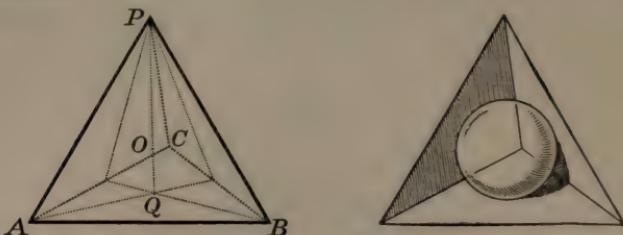
579. Corollary. *The plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

Ex. 1. Prove that the locus of tangents to a sphere from a point outside the sphere is the lateral surface of a cone of revolution.

Ex. 2. If the distance of the point (Ex. 1) from the center of the sphere is equal to the diameter of the sphere, what is the radius of the base of the cone (the locus of the points of tangency)? *Ans.* $r\sqrt{3}/2$.

Proposition 5. Inscribing a Sphere in a Tetrahedron

580. Theorem. *A sphere can be inscribed in any given tetrahedron.*



Given the tetrahedron $P-ABC$.

To prove that a sphere can be inscribed in $P-ABC$.

The plan is to show that the planes bisecting the dihedral \angle meet in a point equidistant from the faces.

Proof. Pass planes through PA and PB bisecting dihedral angles PA and PB and intersecting in PQ .

1. PQ is equidistant from faces PAB and PAC and is equidistant from faces PAB and PBC ; that is, PQ is equidistant from faces PAB , PAC , and PBC .

1. Why?

2. Now pass a plane bisecting dihedral angle AB .
 2. This plane will intersect PQ at some point O .
 3. Hence O is equidistant from the four faces.
 4. $\therefore O$ is the center of a sphere tangent to the four faces, the radius being the \perp distance from each face.

2. Why?

3. Why?

4. § 560.

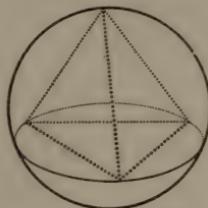
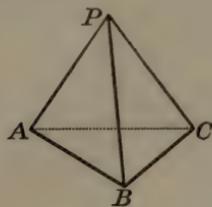
Therefore, a sphere can be inscribed in any given tetrahedron.

Ex. 1. Does the center of a sphere inscribed in a tetrahedron lie in the altitude of the tetrahedron?

Ex. 2. Find the volume of a regular triangular pyramid $D-ABC$, in which $AB = 10$ and the slant height = 12.

Proposition 6. Circumscribing a Sphere

581. Theorem. *A sphere can be circumscribed about any given tetrahedron.*



Given the tetrahedron $P-ABC$.

To prove that a sphere can be circumscribed about it.

The plan is to show that planes that are the perpendicular bisectors of the edges meet in a point equidistant from the vertices. (Can you prove it?)

Proof. Pass planes through the midpoints of AB and BC , \perp to these lines respectively.

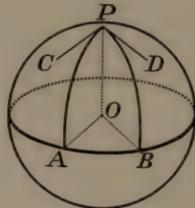
- | | |
|---|---------------------------------|
| 1. These planes intersect in a line (l), every point of which is equidistant from A , B , and C .
Now pass a plane \perp to PA at its midpoint.
2. This plane will intersect line (l) in some one point O equidistant from P , A , B , and C .
3. Hence O is the center of the circumscribed sphere. | 1. Why?
2. Why?
3. § 560. |
|---|---------------------------------|

582. Corollary. *A sphere may be passed through any four points, if each of the points is not in the plane determined by the other three.*

583. Spherical angle. When two arcs of great circles meet on the surface of a sphere, they form a *spherical angle*. This angle is measured by the plane angle formed by the tangents to the circles at their point of intersection. Equal spherical angles can be made to coincide, and conversely.

Proposition 7. Measurement of Spherical Angles

584. Theorem. *An angle formed by arcs of two great circles is measured by an arc of a great circle described from its vertex as a pole and included between its sides, produced if necessary.*



Given PA and PB , arcs of great circles; and AB , the arc of the great circle described with P as its pole and included between PA and PB .

To prove that the spherical $\angle APB$ is measured by arc AB .

The plan is to show that the angle between the tangents to the sides of the angle at P equals the angle between the radii on the great circle whose pole is the vertex of the angle, and hence equals its intercepted arc.

Proof. Draw PC and PD tangent to PA and PB respectively at P . Draw OA , OB , and PO to center O of the sphere.

- | | |
|--|---|
| 1. Then PC is \parallel to OA and PD is \parallel to OB .
2. Hence $\angle CPD = \angle AOB$.
3. But $\angle CPD$ measures the spherical $\angle APB$.
4. And $\angle AOB$ is measured by arc AB .
5. Hence spherical $\angle APB$ is measured by arc AB . | 1. Why?
2. § 446.
3. § 583 and Ax. 5.
4. Why?
5. Why? |
|--|---|

Therefore, an angle formed by arcs . . .

585. Corollary. *A spherical angle has the same measure as the dihedral angle formed by the planes of the two circles.*

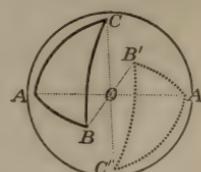
586. Spherical polygons. A spherical polygon is a portion of a sphere bounded by three or more arcs of great circles. The arcs are the *sides* of the polygon, and their points of intersection are the *vertices* of the polygon. A spherical polygon is *convex*, if no side of it when produced enters the polygon.

A diagonal, median, altitude, bisector of an angle, etc., when referring to a spherical polygon are understood to refer to arcs of great circles bearing the same relation to the spherical polygon that lines similarly named bear to plane polygons.

587. The central polyhedral angle of a spherical polygon is the polyhedral angle formed by the intersection of the planes of its sides. Since these planes all intersect at the center of the sphere, the *face angles* of the central polyhedral angle are measured by the sides of the spherical polygon intercepted by them.

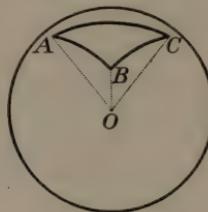
588. Congruent spherical triangles. Two spherical triangles are congruent, if they can be made to coincide. If two spherical triangles are congruent, the sides and angles of one are equal respectively to the sides and angles of the other and are arranged in the same order.

589. Symmetric spherical triangles. If from the vertices A , B , and C of a spherical triangle diameters are drawn to A' , B' , and C' , respectively, the $\triangle A'B'C'$ is said to be symmetric to $\triangle ABC$. If two triangles are symmetric, the sides and angles of one are equal respectively to the sides and angles of the other but are arranged in reverse order.



Proposition 8. Property of a Spherical Triangle

590. Theorem. *Any side of a spherical triangle is less than the sum of the other two sides.*



Given the spherical $\triangle ABC$, with AC its longest side.

To prove that $AC < AB + BC$.

The plan is to refer to the corresponding trihedral angle at the center. (Can you prove it without reading further?)

Proof. Draw OA , OB , and OC .

- | | |
|---|---------------------|
| 1. In trihedral $\angle O-ABC$, face $\angle AOC$
$<$ face $\angle AOB +$ face $\angle BOC$. | 1. § 464. |
| 2. Hence arc $AC < \text{arc } AB + \text{arc } BC$. | 2. § 587 and Ax. 6. |

Therefore, any side of a spherical triangle is less than . . .

591. Symmetric isosceles triangles. Symmetric isosceles spherical triangles are congruent, since the sides and angles of one are equal respectively to the sides and angles of the other in reverse order and hence also in the same order.

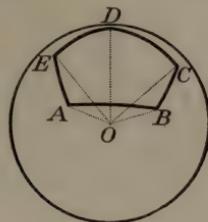
Ex. 1. Suppose that in the figure for Proposition 8, $\angle AOC = 100^\circ$, $\angle AOB = 80^\circ$, and $\angle BOC = 70^\circ$. How many degrees would there be in arc AB ? in arc BC ? in arc AC ?

Ex. 2. Is it possible to have a spherical triangle with sides of 90° , 45° , and 58° ? of 110° , 60° , and 50° ?

Ex. 3. Are two tangents to a sphere from the same point equal?

Proposition 9. Property of a Convex Spherical Polygon

592. Theorem. *The sum of the sides of a convex spherical polygon is less than 360° .*



Given the convex spherical polygon $ABCDE$.

To prove that the sum of the sides AB, BC, CD , etc., $< 360^\circ$.

The plan is to refer to the corresponding polyhedral angle at the center. (Can you prove it without reading further?)

Proof. Connect the vertices A, B, C , etc., with O , the center of the sphere forming the polyhedral angle $\angle O-ABC$.

- | | |
|--|-------------------------------------|
| 1. The sum of the face \triangle about $O < 360^\circ$.
2. Arcs AB, BC, CD , etc., are the measure of the
face \triangle at O .
3. $\therefore AB + BC + CD + \text{(etc.)} < 360^\circ$. | 1. § 465.
2. § 587.
3. Ax. 6. |
|--|-------------------------------------|

Therefore, the sum of the sides

Ex. 1. A geographical mile is defined as $1'$ (one minute) of arc of a great circle of the earth. How many geographical miles are there in the circumference of the earth?

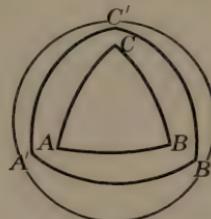
Ex. 2. What is the length in geographic miles of a degree of latitude on the 45th parallel? on the 60th?

Ex. 3. Are all arcs of great circles that are drawn through the pole of a given great circle perpendicular to the circle?

Ex. 4. Can a spherical quadrilateral have quadrants for three of its sides? for four?

Ex. 5. Construct a plane tangent to a sphere at a given point.

593. Polar triangles. The *polar triangle* of a given spherical triangle is formed by the arcs of great circles whose poles are the vertices of the given triangle.



Thus $A'B'$, $B'C'$, $A'C'$ are the arcs of great circles of which C , A , and B are the poles. Hence, $A'B'C'$ is the polar triangle of spherical triangle ABC . If the arcs $A'B'$, $A'C'$, and $B'C'$ were completed, there would be formed other spherical triangles; but the one of these that is the polar triangle of ABC is the one in which A and A' lie on the same side of BC , B and B' on the same side of AC , and C and C' on the same side of AB .

Ex. 1. Is the circle midway between the equator and the north pole equal to half the equator?

Ex. 2. If a spherical triangle is trirectangular, has it a polar triangle?

Ex. 3. How many degrees west of New York (longitude 75° W.) is the 180th meridian? How many degrees east?

Ex. 4. Is the line joining the poles of a circle of a sphere perpendicular to all the radii of the circle?

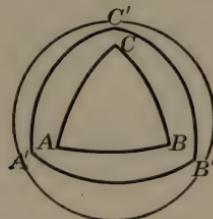
Ex. 5. If two planes are tangent to a sphere at the opposite ends of a diameter, are they parallel?

Ex. 6. If two spheres are tangent to each other, is their common tangent plane perpendicular to the line of centers?

Ex. 7. When it is noon, solar time, at a point on the 40th parallel north latitude at the time of an equinox, what is the elevation of the sun?

Proposition 10. First Property of Polar Triangles

594. Theorem. *If one spherical triangle is the polar triangle of another, then, reciprocally, the second is the polar triangle of the first.*



Given the spherical triangle ABC and its polar triangle $A'B'C'$.

To prove that ABC is the polar triangle of $A'B'C'$.

The plan is to show that A' , B' , and C' are the poles of arcs AB , AC , and BC .

Proof.

- | | |
|--|-----------|
| 1. A is the pole of $B'C'$ and B is the pole of $A'C'$. | 1. § 593. |
| 2. Hence C' is a quadrant's distance from A and B . | 2. § 574. |
| 3. Hence C' is the pole of AB . | 3. § 575. |
| Likewise B' may be shown to be the pole of AC and A' of BC . | |
| 4. $\therefore A'B'C'$ is the polar triangle of $\triangle ABC$. | 4. § 593. |

Therefore, if one spherical triangle

Ex. 1. Can the sum of the sides of a convex spherical polygon be as large as a great circle of the sphere?

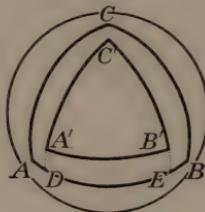
Ex. 2. Are all the great circles drawn through the pole of a circle of a sphere perpendicular to the circle?

Ex. 3. Can a semicircle form one side of a convex spherical triangle?

Ex. 4. If a plane is tangent to a sphere, does it contain all the lines tangent to the sphere at the point of tangency of the plane?

Proposition 11. Second Property of Polar Triangles

595. Theorem. In two polar triangles, any angle of either is the supplement of the opposite side of the other.



Given ABC and $A'B'C'$, any two polar triangles, with side AB of the first, opposite angle C' of the second.

To prove that $AB + \angle C' = 180^\circ$.

The plan is to prove that C' (which $= DE$) $+ AB = 180^\circ$.

Proof. Extend $C'A'$ and $C'B'$ to meet AB in D and E respectively.

- | | |
|---|--|
| 1. A is the pole of $C'B'E$.
2. Hence AE is 90° .
Likewise BD is 90° .
3. Hence $AE + BD = 180^\circ$.
4. That is, $AB + DE = 180^\circ$.
5. But DE is the measure of $\angle C'$.
6. $\therefore AB + \angle C' = 180^\circ$. | 1. § 593.
2. § 574.

3. Ax. 1.
4. Ax. 6.
5. § 584.
6. Ax. 6. |
|---|--|

Therefore, in two polar triangles, any angle

Ex. 1. A spherical triangle has as sides 80° , 100° , and 100° . How many degrees in the angles of its polar?

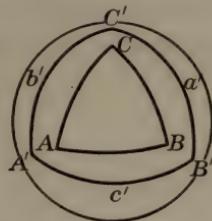
Ex. 2. A spherical triangle has as sides arcs of 70° , 90° , and 95° . How many degrees in the angles of its polar?

Ex. 3. If two angles of a spherical triangle are equal, is its polar isosceles?

Ex. 4. If a spherical triangle is equiangular, is its polar equilateral?

Proposition 12. Angles of a Sphere

596. Theorem. *The sum of the angles of a spherical triangle is greater than 180° and less than 540° .*



Given the spherical $\triangle ABC$.

To prove that $\angle A + \angle B + \angle C > 180^\circ$ and $< 540^\circ$.

Proof. Construct $A'B'C'$, the polar triangle of ABC , with sides a' , b' , and c' , as shown.

- | | |
|---|------------|
| 1. $A + a' = 180^\circ$, $B + b' = 180^\circ$, and $C + c' = 180^\circ$. | 1. § 595. |
| 2. Hence $A + B + C + a' + b' + c' = 540^\circ$. | 2. Ax. 1. |
| 3. But $a' + b' + c' < 360^\circ$. | 3. § 592. |
| 4. Hence $A + B + C > 180^\circ$. | 4. Ax. 12. |
| Since a' , b' , and c' are each $> 0^\circ$, | |
| 5. $a' + b' + c' > 0^\circ$. | 5. Why? |
| 6. $\therefore A + B + C < 540^\circ$. | 6. Why? |

Therefore, the sum of the angles

Ex. 1. Can a spherical triangle have two or even three right angles? obtuse angles?

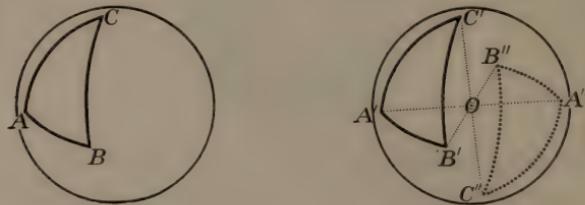
Ex. 2. Is the exterior angle of a convex spherical triangle equal to the sum of the two opposite interior angles?

Ex. 3. What relation obtains between the measure of a spherical angle and that of the dihedral angle formed by the plane of its sides?

Ex. 4. If you know that a spherical quadrilateral is equilateral, can you find the size of one of its angles? Can you if you know it is equiangular?

Proposition 13. Congruent and Symmetric Triangles

597. Theorem. *Two spherical triangles on the same sphere or on equal spheres are either congruent or symmetric if two sides and the included angle of one equal respectively two sides and the included angle of the other.*



Given the spherical $\triangle ABC$ and $\triangle A'B'C'$ on equal spheres and in which $AC = A'C'$, $BC = B'C'$, and $\angle C = \angle C'$ and arranged in the same order, and spherical $\triangle ABC$ and $\triangle A''B''C''$ in which $AC = A''C''$, $BC = B''C''$, and $\angle C = \angle C''$ and arranged in the reverse order.

To prove (1) that $\triangle ABC \cong \triangle A'B'C'$, and (2) that $\triangle ABC$ is symmetric to $\triangle A''B''C''$.

The plan is to show that ABC and $A'B'C'$ can be made to coincide and that when they do $\triangle ABC$ is symmetric to $\triangle A''B''C''$ by definition.

Proof of 1. Place $\triangle ABC$ on $\triangle A'B'C'$ so that $\angle C$ will coincide with its equal $\angle C'$.

- | | |
|--|-------------------------------------|
| 1. CA and CB will coincide respectively with $C'A'$ and $C'B'$; that is, A will fall on A' and B on B' .
2. Hence AB will coincide with $A'B'$.
3. $\therefore \triangle ABC \cong \triangle A'B'C'$. | 1. § 562.
2. § 563.
3. § 588. |
|--|-------------------------------------|

Proof of 2. Since $\triangle ABC$ now coincides with $\triangle A'B'C'$,

- | | |
|---|---|
| 4. $\angle A$, B , and C are equal respectively to $\angle A''$, B'' , and C'' and AB , AC , and BC are equal respectively to $A''B''$, $A''C''$, and $B''C''$ and are arranged in reverse order. | 4. Ax. 6. |
| 5. $\therefore \triangle ABC$ and $\triangle A''B''C''$ are symmetric. | 5. By definition of symmetric \triangle . |

Therefore, two spherical triangles on the same sphere . . .

Ex. 1. Can you make two plane triangles coincide, if their parts are respectively equal, but arranged in the reverse order? two spherical triangles?

Ex. 2. Does the radius drawn to the midpoint of an arc of a sphere bisect the chord of the arc?

Ex. 3. Are two circles of a sphere great circles if their diameters are perpendicular to each other?

Ex. 4. If two spheres intersect, do they intersect in a straight line?

Ex. 5. If two sections of a sphere are made by parallel planes, do they have the same poles?

Ex. 6. Do the planes perpendicular to the edges of a tetrahedron at their midpoints meet in a point?

Ex. 7. If two spherical triangles have two sides and the included angle of one equal to two sides and the included angle of the other, are they congruent triangles?

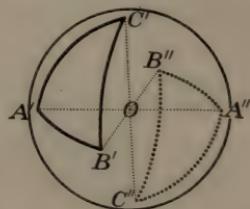
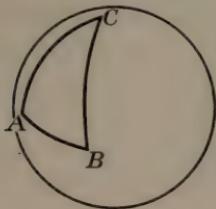
Ex. 8. What is the locus of a point at a given distance from a given sphere? May there be more than one part to the locus?

Ex. 9. What is the locus of a point equidistant from the centers of two spheres? May there be more than one part to the locus?

Ex. 10. What is the locus of a point equidistant from two parallel planes and at a given distance from a given point between the parallel planes? Under what conditions is there no locus?

Proposition 14. Congruent and Symmetric Triangles

598. Theorem. *Two spherical triangles on the same sphere or on equal spheres are either congruent or symmetric, if a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.*



Given the spherical $\triangle ABC$ and $\triangle A'B'C'$, in which $AB = A'B'$, $\angle A = \angle A'$, and $\angle B = \angle B'$, and arranged in the same order; and spherical $\triangle ABC$ and $\triangle A''B''C''$, in which $AB = A''B''$, $\angle A = \angle A''$, and $\angle B = \angle B''$, but arranged in the reverse order.

To prove (1) that $\triangle ABC \cong \triangle A'B'C'$ and (2) that $\triangle ABC$ is symmetric to $\triangle A''B''C''$.

Proof of 1. (Left to the student.)

SUGGESTIONS. Place $\triangle ABC$ on $\triangle A'B'C'$ so that AB coincides with its equal $A'B'$, and so that AC will fall along $A'C'$. Show that BC will fall along $B'C'$ (§ 583) and that C and C' coincide (§ 563).

Proof of 2. (Left to the student.)

SUGGESTION. See Proof of 2 in Proposition 13.

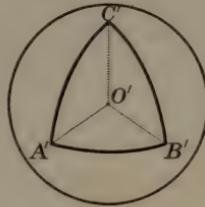
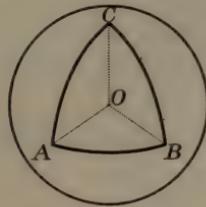
Ex. 1. What is the locus of the midpoints of all the equal chords of a sphere?

Ex. 2. What is the locus of the center of a sphere of given radius tangent to a given plane? a given line?

Ex. 3. What is the locus of the center of a sphere of given radius tangent to a given sphere?

Proposition 15. Congruent and Symmetric Triangles

599. Theorem. *Two spherical triangles on the same sphere, or on equal spheres, are either congruent or symmetric if the three sides of one are equal respectively to the three sides of the other.*



Given, on equal spheres, the spherical $\triangle ABC$ and $A'B'C'$, in which $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$.

To prove that $\triangle ABC$ and $\triangle A'B'C'$ are either congruent or symmetric.

The plan is to show that $\triangle ABC$ and $A'B'C'$ are also mutually equiangular and are therefore either equal or symmetric by § 597 or § 598.

Proof. Join A , B , C , A' , B' , and C' , with the centers of their respective spheres forming trihedral $\angle O$ and O' .

- | | |
|---|-----------------------|
| 1. AB , AC , and BC are the measures of the three face \angle of trihedral $\angle O$, and $A'B'$, $A'C'$, and $B'C'$ are the measures of the three face \angle of trihedral $\angle O'$. | 1. § 587. |
| 2. Hence the face \angle of trihedral $\angle O$ and O' are respectively equal. | 2. Why? |
| 3. Hence the corresponding dihedral \angle are equal. | 3. § 468. |
| 4. Hence the corresponding angles of the spherical triangles are equal. | 4. § 585. |
| 5. $\therefore \triangle ABC$ and $\triangle A'B'C'$ are either congruent or symmetric. | 5. § 597
or § 598. |

Therefore, two spherical triangles

X

600. Corollary. *Two spherical triangles on the same sphere or on equal spheres are either congruent or symmetric, if the three angles of one are equal respectively to the three angles of the other.*

Draw the polar Δ of each. These polar Δ will be mutually equilateral by § 595. They are therefore congruent. Hence they are mutually equiangular. Therefore their polar Δ are mutually equilateral and hence congruent or symmetric, by § 599.

601. Corollary. *If two sides of a spherical triangle are equal, the angles opposite these sides are equal.*

Join the vertex angle with the midpoint of the opposite side. This forms two triangles mutually equilateral, hence mutually equiangular by § 599.

602. Corollary. *If two angles of a spherical triangle are equal, the sides opposite these angles are equal.*

Construct the polar triangle of the given triangle. This will be isosceles by § 601. Hence the angles opposite these equal sides will be equal. Hence the sides of the given triangle opposite these angles will be equal. Why?

603. Corollary. *If a spherical triangle is equilateral, it is also equiangular, and conversely.*

Ex. 1. If a spherical quadrilateral is equilateral, do the diagonals bisect each other? Is the converse true?

Ex. 2. Does the arc that bisects the vertex angle of an isosceles spherical triangle bisect the opposite side?

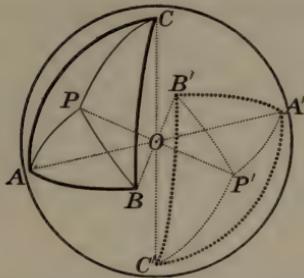
Ex. 3. If a ball is free to roll about on the top of a table, what is the locus of its center?

Ex. 4. Are the arcs bisecting the base angles of an isosceles spherical triangle and terminating in the opposite sides equal?

Ex. 5. If two parallel arcs are cut by the arc of a great circle of a sphere, are the alternate interior angles equal?

Proposition 16. Symmetric Triangles Equivalent

604. Theorem. Two symmetric spherical triangles are equivalent.



Given the symmetric spherical $\triangle ABC$ and $A'B'C'$.

To prove that $\triangle ABC$ is equivalent to $\triangle A'B'C'$.

The plan is to divide each triangle into three isosceles triangles and show that pairs of these are congruent.

Proof. Let P be the pole of the \odot through A, B , and C , and draw the diameter POP' . With arcs of great circles, join P with A, B , and C , and P' with A', B' , and C' .

- | | |
|---|-----------|
| 1. $\triangle PAB$ and $P'A'B'$ are symmetric. | 1. § 589. |
| 2. But $PA = PB$. | 2. § 573. |
| 3. Hence $\angle POA = \angle POB$. | 3. Why? |
| 4. Hence $\angle P'OA' = \angle P'OB'$. | 4. Why? |
| 5. Hence $P'A' = P'B'$; that is, $\triangle PAB$ and $P'A'B'$ are isosceles symmetric triangles. | 5. Why? |
| 6. Hence $\triangle PAB \cong \triangle P'A'B'$. | 6. § 591. |
| Likewise $PAC \cong P'A'C'$, and $PBC \cong P'B'C'$. | |
| 7. $\therefore ABC$ is equivalent to $A'B'C'$. | 7. Ax. 1. |

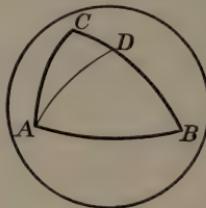
Ex. 1. Is the converse of Proposition 16 true?

Ex. 2. Is the bisector of the vertex angle of an isosceles spherical triangle perpendicular to the base?

Ex. 3. Is one fourth of a circle of a sphere a quadrant?

Proposition 17. Triangles with Unequal Angles

605. Theorem. *If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal and the side opposite the greater angle is the greater.*



Given the spherical $\triangle ABC$ with $\angle A > \angle B$.

To prove that $BC > AC$.

Proof. Draw arc AD of a great circle, making $\angle DAB = \angle B$.

- | | |
|---|--|
| 1. $AD = BD$.
2. But $CD + AD > AC$.
3. Hence $CD + DB > AC$.
4. $\therefore BC > AC$. | 1. § 602.
2. § 590.
3. Why?
4. Ax. 6. |
|---|--|

Therefore, if two angles of a spherical

606. Corollary. *If two sides of a spherical triangle are unequal, the angles opposite these sides are unequal and the angle opposite the greater side is the greater.*

Use the indirect method analogous to § 173.

Ex. 1. Is a spherical triangle a spherical polygon?

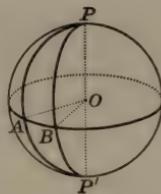
Ex. 2. If you know the size of two angles of a spherical triangle, can you find the size of the third angle?

Ex. 3. Two boards are nailed at right angles to each other, forming a trough. The plane bisecting the dihedral angle so formed is vertical. A sphere 8 inches in diameter is placed in the trough. How far from the edge of the dihedral angle will the ball be tangent?

607. A **lune** is the portion of a sphere included between two semicircles of great circles of the sphere.

Thus $PAP'B$ is a lune.

608. The **angle of a lune** is the angle between the semicircles. Thus $\angle APB$ or $\angle AP'B$ is the angle of lune $PAP'B$. A lune of 180° is, of course, a hemisphere.



609. The **unit of measure** of spherical surfaces and polygons is either a square inch, square foot, etc., or a spherical degree.

610. The **spherical degree** is the surface bounded by two quadrants and an arc of 1° . There are, then, 360 spherical degrees in the surface of a hemisphere, or 720 spherical degrees in the surface of the sphere. Since a spherical degree is one half a lune of one degree, the area of a lune is twice the number of degrees in its angle.

611. The **spherical excess** of a spherical triangle is the number of degrees the sum of its angles is in excess of 180° . The spherical excess of a spherical polygon is equal to the number of degrees that the sum of its angles is in excess of $(n - 2) \times 180^\circ$, since the polygon may be cut into $(n - 2)$ triangles by arcs of great circles from a vertex to all the non-adjacent vertices.

Ex. 1. If the opposite sides of a spherical quadrilateral are equal, are the opposite angles equal?

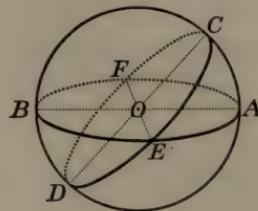
Ex. 2. If two parallel planes are each tangent to a sphere, is the line connecting their points of tangency a diameter?

Ex. 3. How many degrees are there in an angle of an equiangular spherical triangle?

Ex. 4. Three equal spheres are mutually tangent. Could a fourth sphere of the same size be tangent to each of the other three?

Proposition 18. Area of Triangle and Lune

612. Theorem. *If two semicircles of great circles intersect on the surface of a hemisphere, the sum of the areas of the two opposite spherical triangles thus formed is equal to the area of a lune whose angle is equal to the angle between the semicircles.*



Given semicircles AEB and CED intersecting at E on the surface of hemisphere $E-ACBD$ of sphere O , forming the opposite $\triangle AEC$ and BED .

To prove that $\triangle AEC + \triangle BED =$ a lune whose angle equals $\angle AEC$.

The plan is to show that the triangle symmetric to $\triangle BED$ with $\triangle AEC$ form the lune.

Proof. Complete the sphere and continue the arcs AEB and CED intersecting at F .

- | | |
|---|-------------------------------------|
| 1. $\triangle BED$ is symmetric to $\triangle AFC$.
2. Hence $\triangle BED = \triangle AFC$.
3. $\therefore \triangle AEC + \triangle BED =$ lune $EAFC$, whose angle is $\angle AEC$. | 1. § 589.
2. § 604.
3. Ax. 6. |
|---|-------------------------------------|

Therefore, if two semicircles of great circles intersect

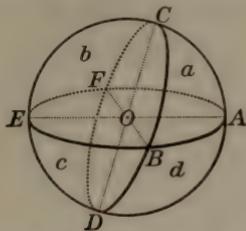
Ex. 1. Is a lune a portion of the volume of a sphere?

Ex. 2. Are two lunes that have the same diameter equivalent?

Ex. 3. What is the area in spherical degrees of a lune whose angle is 60° ?

Proposition 19. Area of Spherical Triangle

613. Theorem. A spherical triangle is equivalent to a lune whose angle is half the spherical excess of the triangle.



Given spherical $\triangle ABC$ on a sphere whose surface is S .

To prove that $\triangle ABC =$ a lune whose \angle is $\frac{1}{2}(\angle A + \angle B + \angle C - 180^\circ)$.

Proof. Complete arcs AB , AC , and BC . Represent $\triangle ABC$, CBE , EBD , and DBA as $\triangle a$, b , c , and d , respectively, as shown.

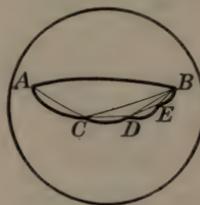
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|--|--------------------|
| 1. $\triangle a + \triangle b =$ lune $\angle A$ (that is, the lune whose angle is $\angle A$). | 1. § 607. |
| 2. Hence $\triangle b =$ lune $\angle A - \triangle a$. | 2. Ax. 2. |
| 3. Similarly $\triangle d =$ lune $\angle C - \triangle a$. | 3. Ax. 2. |
| 4. Also $\triangle a + \triangle c =$ lune $\angle B$. | 4. § 612. |
| Adding equations 2, 3, and 4, | |
| 5. $\triangle a + \triangle b + \triangle c + \triangle d =$ lune $\angle A +$ lune $\angle B +$ lune $\angle C - 2\triangle a$. | 5. Ax. 1. |
| 6. Now $\triangle a + \triangle b + \triangle c + \triangle d = \frac{1}{2}S =$ lune $\angle 180^\circ$.
Substituting in equation 5 and transposing, | 6. § 608. |
| 7. $2\triangle a =$ lune $\angle A +$ lune $\angle B +$ lune $\angle C -$ lune $\angle 180^\circ$. | 7. Ax. 6
and 2. |
| 8. $\therefore \triangle a = \frac{1}{2}$ lune $(\angle A + \angle B + \angle C - 180^\circ)$. | 8. Ax. 4. |

614. Corollary. A convex spherical polygon is equivalent to a lune whose angle is half the spherical excess of the polygon.

Draw the diagonals from any vertex and thus divide the polygon into $(n - 2)$ triangles and apply Theorem 613.

Proposition' 20. Shortest Distance on a Sphere

615. Theorem. *The shortest line that can be drawn on the surface of a sphere, connecting two points on the sphere, is the minor arc of a great circle through the two points.*



Given two points A and B on a sphere, joined by the arc AB of a great circle.

To prove that arc AB is the shortest distance on the sphere between A and B .

The plan is to show that any line joining A and B other than the arc AB of a great circle is either equal to or greater than some line that is known to be greater than arc AB .

Proof. Let $ACDEB$ be any other line on the sphere joining A and B . Let C be a point on $ACDEB$ not on arc AB . Draw arcs AC and BC of great circles.

- | | |
|--|--|
| <ol style="list-style-type: none"> 1. Arc $AB < \text{arc } AC + \text{arc } BC$. 2. Now obviously either $ACDEB$ coincides with broken line ACB or it does not. If it does, then $ACDEB = ACB > AB$, and the theorem is proved. If not, then some point D can be found that is not on AC or BC. Assume D to be on irregular line CB. Draw arcs CD and DB of great circles. Then $CD + DB > CB$. 3. Hence $AC + (CD + DB) > AC + CB$. That is, $ACDB > ACB > AB$. | <ol style="list-style-type: none"> 1. § 590. 2. § 590. 3. Ax. 10. |
|--|--|

By continuing this process, it may be shown that any line other than arc AB of a great circle is either equal to or greater than some line that is known to be greater than AB . Therefore arc AB is the shortest distance on the sphere between A and B .

If arc AB of a great circle is a major arc, then of course the corresponding minor arc AB is shorter.

Therefore, the shortest line



EXERCISES

Ex. 1. How many spherical degrees are there in the area of a spherical triangle whose angles are 80° , 120° , and 100° ? What part of the surface of the sphere is this?

Ex. 2. How many spherical degrees are there in a trirectangular triangle? What part of the surface of the sphere is this?

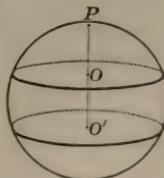
Ex. 3. How many spherical degrees are there in a hemisphere?

Ex. 4. If the spherical excess of a spherical triangle is n° , what part of the sphere does it cover?

Ex. 5. Represent a spherical degree on the surface of a hemisphere.

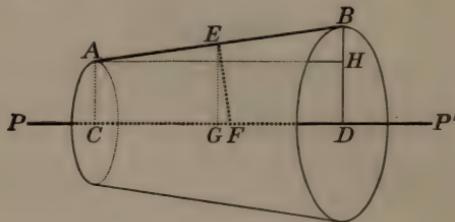
Ex. 6. What is the area of a spherical polygon whose angles are 120° , 140° , 160° , 150° , 160° ? What part of the sphere does it cover?

616. A *zone* is the portion of a sphere included between two parallel planes. The sections of the sphere made by the planes are the *bases*. The perpendicular distance between the planes of the bases is the *altitude* of the zone. If one of the parallel planes is tangent to the sphere, a *zone of one base* is formed. PO is the altitude of the zone of one base and OO' the altitude of the second zone.



Proposition 21. Area Generated by Revolving Line

617. Theorem. *The area of the surface generated by the revolution of a straight line segment about an axis in its plane, but not crossing it, is equal to the projection of the segment upon the axis multiplied by the length of the circle whose radius is a perpendicular erected at the midpoint of the segment and terminated by the axis.*



Given a line AB revolving about axis PP' in its plane, generating area a , CD the projection of AB on PP' , E the midpoint of AB , $EF \perp$ to AB , F lying in PP' .

To prove that area $a = CD \times 2\pi EF$.

The plan is to express first the area in terms of the slant height and circumference of the midsection; then, by reference to similar triangles, to derive a new product and substitute it for its equivalent in the equation.

Proof. Draw $EG \perp$ to CD , and $AH \perp$ to BD .

- | | |
|---|---------------|
| 1. Since AB generates the frustum of a cone, area $a = AB \times 2\pi EG$
$= 2\pi(AB \times EG)$. | 1. § 549. |
| 2. But $\triangle GEF \sim \triangle HAB$. | 2. Why? |
| 3. Hence $\frac{AB}{AH} = \frac{EF}{EG}$. | 3. Why? |
| 4. So $AH \times EF = AB \times EG$.
Substituting in Eq. 1, | 4. Why? |
| 5. Area $a = 2\pi(AH \times EF)$. | 5. Subst. Ax. |

6. But $CD = AH$.
 7. \therefore area $a = CD \times 2\pi EF$.

6. Why?
 7. Subst. Ax.

Now if a' and $E'F'$ represent the limiting values of a and EF as A approaches C , that is, if $a \rightarrow a'$, and $EF \rightarrow E'F'$,

8. $CD \times 2\pi EF \rightarrow CD \times 2\pi E'F'$. 8. § 517, 2.
 Remembering that a always equals
 $CD \times 2\pi EF$,
 9. $a' = CD \times 2\pi E'F'$. 9. § 517, 1.

Hence the theorem is true for the special case in which A lies on PP' .

Similarly it may be shown by the method of limits that the theorem is true for the special case in which AB is parallel to PP' .

Therefore, the area of the surface

Ex. 1. What type figure is generated by the limiting value of a as $a \rightarrow a'$?

Ex. 2. AB is revolved about BC as its axis. Angle ABC is 30° , and AB is $20''$ long. EC is perpendicular to AB at its midpoint and meets BC at C . Find the length of EC and of BC . Find the area of the surface generated.

Ex. 3. Find the area of the surface of the cone of Exercise 2 by finding the radius of its base and substituting in the formula $S = \pi rl$. Is your answer the same as that of Exercise 2?

Ex. 4. What type figure is generated by AB in Proposition 21, when AB is parallel to PP' ?

Ex. 5. When AB is parallel to PP' , what is the projection of AB on PP' ? Where, then, does EF meet PP' ?

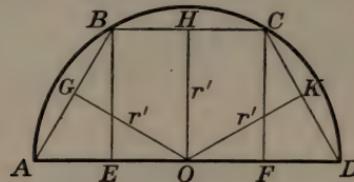
Ex. 6. In trapezoid $ABCD$, $\angle A$ and $\angle B$ are right angles. $BC = 12''$, $CD = 24''$, and $DA = 8''$. What area is generated by the revolution of CD about AB ?

Ex. 7. If AB crosses PP' , what is the nature of the surface generated by revolving AB about PP' ?

Ex. 8. AB is 20 in. long. PP' bisects AB at D , making angle PDA equal 30° . AB is revolved about PP' . What is the area of the surface generated?

Proposition 22. Area of Sphere

618. Theorem. *The area of a sphere is equal to the product of its diameter by the length of one of its great circles.*



Given a sphere generated by the revolution of the semi-circle $ABCD$ about its diameter AD ; its center, O ; its surface, S ; its radius, r ; and its diameter, d .

To prove that $S = d \times 2\pi r$.

The plan is to inscribe half of a regular polygon in the hemisphere and revolve the entire figure about AD and prove that the surface generated by the polygon = 2π times the apothem OG , and apply the theory of limits.

Proof. Inscribe in the semicircle half of a regular polygon of an even number of sides, $ABCD$, with apothem r' . Draw BE and $CF \perp$ to AD and from O draw $OG \perp$ to AB , $OH \perp$ to BC , and $OK \perp$ to CD .

1. OG , OH , and OK bisect AB , BC , and CD respectively.
2. $OG = OH = OK$.

Let r' represent the length of these lines.

Let the entire figure revolve about AD .

3. Area generated by $AB = AE \times 2\pi r'$.
4. Area generated by $BC = EF \times 2\pi r'$.
5. Area generated by $CD = FD \times 2\pi r'$.

Let S' represent the area generated by the semipolygon.

6. $S' = (AE + EF + FD) \times 2\pi r'$.
7. $S' = d \times 2\pi r'$.

1. Why?

2. § 248.

3. § 617.

4. § 617.

5. § 617.

6. Ax. 1.

7. Ax. 6.

Now let the number of sides of the polygon be continuously increased.

8. $S' \rightarrow S, r' \rightarrow r.$
9. Hence $d \times 2\pi r' \rightarrow d \times 2\pi r.$
10. $\therefore S = d \times 2\pi r.$

8. § 385.
9. § 517, 2.
10. § 517, 1.

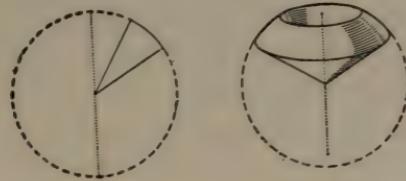
Therefore, the area of a sphere . . .

619. Corollary. *The area of a sphere is equal to the area of four great circles (that is, area of sphere = $4\pi r^2$).*

620. Corollary. *The area of a zone is equal to the altitude of the zone multiplied by the length of a great circle.*

In § 618, show by the method of limits that the zone generated by $AB = AE \times 2\pi r$; by $BC = EF \times 2\pi r$; etc.

621. A **spherical sector** is the solid generated by the revolution of a sector of a circle about one of the diameters of the circle. The zone generated by the arc of the sector is called the *base* of the spherical sector.



622. A spherical segment is the volume of the sphere included between two parallel planes. If one of the planes is tangent to the sphere, the segment is called a *spherical segment of one base*.

623. A spherical pyramid is the solid bounded by a spherical polygon and the planes of its sides meeting at the center of the sphere. The *vertex* of the pyramid is the center of the sphere.

624. The **volume of a sphere** is the number of cubic units within its surface.

625. If a cube is circumscribed about a sphere and the center of the sphere is joined to all the vertices of the cube, six regular congruent pyramids will be formed. Now if tangent planes are passed through the points where the edges of these pyramids cut the sphere and this process be continuously repeated, the resulting polyhedrons will have areas and volumes the sums of which will approach as limits the area and volume respectively of the sphere.

Ex. 1. Are equal chords of a circle of a sphere equidistant from a pole of the circle?

Ex. 2. Two equal spheres whose radii are r intersect so that each passes through the center of the other. What is the area of the circle of intersection?

Ex. 3. The angles of a spherical triangle are 80° , 100° , and 110° . What is the angle of the equivalent lune?

Ex. 4. Find each angle of the equilateral spherical triangle that is equivalent to the spherical triangle of Exercise 3.

Ex. 5. Prove that the areas of two spheres are to each other as the squares of their radii.

Ex. 6. Prove that the areas of two zones on the same sphere are to each other as their altitudes.

Ex. 7. Solve $S = 4\pi r^2$ for r . What is r when S equals 55.44? (Use $\frac{22}{7}$ for π .)

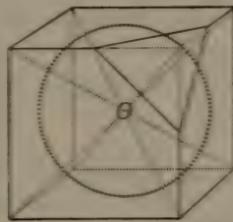
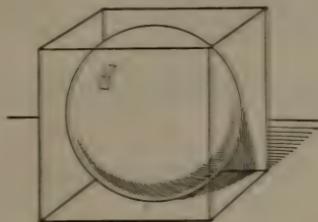
Ex. 8. If two lines are tangent to a sphere from the same point, does their plane pass through the center of the sphere?

Ex. 9. Considering the earth as a sphere with a radius of 3960 mi., how far is it from the 49th parallel north latitude to the north pole?

Ex. 10. In Exercise 2, page 421, what angle do the tangents make with the line joining the external point to the center of the sphere?

Proposition 23. The Volume of a Sphere

626. Theorem. *The volume of a sphere is equal to the product of the area of its surface and one third of its radius (that is, volume, $V = \frac{4}{3} \pi r^3$).*



Given the sphere O , with volume V and radius r .

To prove that $V = \frac{4}{3} \pi r^3$.

The plan is to circumscribe a cube about the sphere, connect its vertices with the center, prove that the sum of the volumes of the pyramids formed equals the sum of their bases $\times \frac{1}{3}r$, then increase the number of sides and apply the theory of limits.

Proof. Circumscribe about the sphere a cube with surface S and volume V' and connect O with each vertex. There will be formed six congruent regular square pyramids, the sum of whose volumes equals V' and whose altitude equals r .

$$1. V' = S \times \frac{1}{3}r.$$

Now continuously increase the number of faces by passing planes tangent to the sphere where the several edges of the pyramids cut the sphere.

Letting V' represent the varying sum of the volumes of the pyramids and S the surface of the varying polyhedron,

$$2. V' \text{ always } = S \times \frac{1}{3}r.$$

$$3. V' \rightarrow V, \text{ and } S \rightarrow 4\pi r^2.$$

$$4. \text{ Hence } S \times \frac{1}{3}r \rightarrow 4\pi r^2 \times \frac{1}{3}r \\ \rightarrow \frac{4}{3}\pi r^3.$$

$$5. \therefore V = \frac{4}{3}\pi r^3.$$

1. § 520, Ax. 1.

2. Why?

3. § 625.

4. § 517, 2.

5. § 517, 1.

NUMERICAL EXERCISES(Use $\pi = \frac{22}{7}$.)

Ex. 1. What are the volume and the area of a sphere whose radius is 8"? of a sphere whose diameter is 15"?

Ex. 2. What is the radius of the sphere whose volume is $821\frac{1}{3}$ cu. in.?

Ex. 3. What are the volume and the surface of a volley ball $8\frac{1}{4}$ " in diameter?

Ex. 4. Compare the surfaces of two spheres in which $\frac{r}{r'} = \frac{1}{2}$.
Compare their volumes.

Ex. 5. Compare the radii of two spheres in which $\frac{S}{S'} = \frac{1}{4}$.
Compare their volumes.

Ex. 6. If the radius of a sphere is 8", what is the area of a lune whose angle is 36° ?

Ex. 7. What is the angle of elevation of the sun at noon on December 21 at a place on the 42d parallel, north latitude?

Ex. 8. If a sphere is inscribed in a cube, what is the ratio of the volume of the sphere to that of the cube?

Ex. 9. The area of a given lune is $\frac{1}{20}$ that of the sphere.
How many degrees are there in the angle of the lune?

Ex. 10. Find the area of a triangle on the surface of the earth, the sum of whose angles is 200° . ($d = 7920$ mi.)

Ex. 11. A ball bearing for a certain car is 1.85 cm. in diameter, if steel has a density of 7.82 grams per cubic centimeter. What is its weight in grams?

Ex. 12. If one sphere has a volume double that of another, how do their areas compare?

Ex. 13. If a sphere is inscribed in a cylinder of revolution (that is, is tangent to its elements and bases), how does the surface of the sphere compare with the lateral surface of the cylinder?

Ex. 14. The diameter of one orange is 25% greater than that of another. Its volume is ____ % greater.

Ex. 15. Find the area of the earth's surface. ($r = 3960$ mi.)

Ex. 16. The radius of a given sphere is $10''$. Find the area of a circle $4''$ from the center.

Ex. 17. Find the area of the spherical triangles having the following angles and radii (use $\pi = \frac{22}{7}$):

(a) $108^\circ, 96^\circ, 80^\circ$ on a sphere $r = 8''$.

(b) $90^\circ, 75^\circ, 85^\circ$ on a sphere $r = 10''$.

(c) $60^\circ, 112^\circ, 80^\circ$ on a sphere $r = 36''$.

Ex. 18. The sides of a given spherical triangle are $88^\circ, 82^\circ$, and 100° . Find the area of its polar triangle on a sphere whose radius is $20''$.

Ex. 19. What is the area of a trirectangular spherical triangle on a sphere whose radius is $12''$?

Ex. 20. Why do ships, in going from New York to a European port due east, travel first a little north of east, then finally a little south of east?

Ex. 21. When it is noon June 21 on the 49th parallel, north latitude, what is the elevation of the sun?

Ex. 22. Given $V = \frac{4}{3} \pi r^3$. Solve for r .

Ex. 23. Evaluate $V = \frac{1}{3} h(B + b + \sqrt{Bb})$ for V when $h = 10$, $B = 25$, and $b = 15$. (Volume of frustum of pyramid.)

Ex. 24. Evaluate $V = \frac{1}{3} h(R^2 + r^2 + Rr)$ for V when $h = 5$, $R = 8$, and $r = 4$. (Volume of frustum of circular cone.)

Ex. 25. A cylinder has a radius equal to the radius of a sphere and an altitude equal to the diameter of the sphere. Find the ratio of their volumes.

Ex. 26. A cone has a radius equal to the radius of a sphere and an altitude equal to the diameter of the sphere. What is the ratio of their volumes?

Ex. 27. Evaluate $S = \pi l(R + r)$ for S when $l = 8$, $R = 12$, and $r = 10$. (Lateral area of frustum of cone of revolution.)

Ex. 28. Evaluate $V = \frac{1}{3} rS$ for V when $r = 12$ and $S = 28$. (Volume of spherical sector.)

MISCELLANEOUS EXERCISES

Ex. 1. If two sections of a sphere made by planes are equal, are they equally distant from the center?

HINT. Are $\triangle OAB$ and OCD congruent?

Ex. 2. If two sections of a sphere made by planes are unequal, is the one more remote from the center the smaller?

HINT. See § 251.

Ex. 3. Can you bisect a spherical angle?

HINT. See Exercise 2, page 51, of *Plane Geometry*.

Ex. 4. Can you determine the poles of a great circle through two given points A and B on a sphere?

HINT. Does the pole lie in the plane perpendicular to OA at O ? In what other plane does it lie?

Ex. 5. Can you draw a circle through three given points on a sphere?

HINT. What is the locus of points on the sphere equidistant from A and B ? from A and C ?

Ex. 6. If two spheres are tangent, does the line of centers, OO' , pass through the point, P , of tangency?

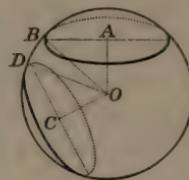
HINT. Assume for the moment that OO' does not pass through P but meets the surface of sphere O at Q . Is QO equal to PO ?

Ex. 7. If two arcs of great circles intersect on a sphere, are the vertical angles equal?

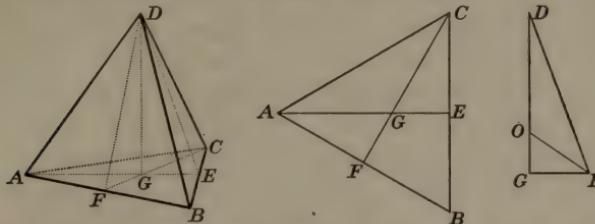
HINT. If the tangents to the arcs are produced, do the vertical plane angles measure the vertical spherical angles?

Ex. 8. If two spheres intersect, is the plane of the intersection perpendicular to the line joining the centers?

HINT. Pass a plane through OO' . Its intersection with the spheres consists of two intersecting circles. Is the common chord of these circles perpendicular to OO' ? See § 415.



Ex. 9. Can you prove that the center of a sphere inscribed in a regular tetrahedron whose edge is a lies on the altitude one fourth the distance from the base to the vertex?



HINTS. (1) Let DG be the altitude. Can you show that G is the center of $\triangle ABC$?

(2) Can you show that G is equidistant from faces ABD , BCD , and ACD ?

(3) Does the center of the inscribed sphere lie on DG ? Why?

(4) Since it is equidistant from the sides of dihedral $\angle BC$, it lies on the bisector of plane $\angle \text{ } \underline{\quad}$.

(5) Let the bisector of $\angle DEG$ intersect DG at O . The ratio of GO to OD equals the ratio of GE to $\underline{\quad}$.

(6) Does $DE = AE$? What is the ratio of GE to AE ? of GE to DE ? of GO to DO ? of GO to GD ?

Ex. 10. What is the volume of the tetrahedron in Exercise 9 in terms of a ?

Ex. 11. A sphere and a cone rest on the same horizontal plane. The radius of the sphere is the same as that of the cone, and the altitude of the cone is equal to the diameter of the sphere. How far from the given plane is a plane parallel to it that makes equal sections in the sphere and the cone? *Ans.* $x = 2r/5$.

Ex. 12. If the sphere in Exercise 11 is made to touch the cone, how far apart are the point of support of the sphere and the center of the base of the cone (in terms of the radius of the sphere)? *Ans.* $(r + r\sqrt{5})/2$.

Ex. 13. Two spheres, one 4" and the other 8" in diameter, rest on the same table in contact with each other. How far apart are the points of support?

INSTRUCTIONAL TESTS

TEST IX (Book Six)

DIRECTIONS. Write the numbers of the exercises in a column. Opposite each number answer the question, as follows: If the answer is "Yes — always," write a plus sign (+). If the answer is "No — never," write a minus sign (-). If the answer is "It may be (one cannot tell)," write a zero (0). If you cannot readily decide upon the answer to a question, go on to the next and try it again later. Your score is the number of questions answered correctly. *Time limit, forty minutes.*

- + 1. A point A not in plane MN is joined to points B and C in MN such that $AB = AC$. Do AB and AC make equal angles with MN ?
- 0 2. AB is \perp to CD . Is $AB \perp$ to plane MN containing CD ?
- 3. Can a line be \perp to each of two intersecting planes?
4. AB cannot meet CD . Is $AB \parallel$ to CD ?
5. Plane MN is \parallel to one of two intersecting lines. Is it \parallel to the other?
6. Plane MN contains one of two \parallel lines. Is it \parallel to the other?
7. Can two planes have only two points in common?
8. Parallel lines AB and CD cut a plane. Do they make equal angles with their projections on this plane?
- 0 9. Two given \triangle have their sides \parallel each to each. Are they equal?
10. Does one point not in plane MN determine a plane parallel to plane MN ?
11. Each of two given lines is \perp to the edge of a dihedral angle. Do they form the plane angle of the dihedral angle?
- + 12. AB is \perp to plane MN . Plane PQ contains AB . Is $PQ \perp$ to MN ?
13. Plane MN is \perp to both faces of a dihedral angle. Is it \perp to the edge of the dihedral angle?
- 14. AB is \perp to CD . Is every plane through $AB \perp$ to CD ?

15. Do a line and its projection on a given plane determine a plane \perp to the given plane?
16. Do the three points A , B , and C determine a plane?
17. Can a trihedral angle have as face angles, angles of 70° , 40° , and 20° , respectively?
18. Can a convex polyhedral angle have as face angles 80° , 100° , 90° , 80° , 50° ?
19. Two planes, MN and PQ , intersect a dihedral angle, the intersections forming two equal angles. Is $MN \parallel$ to PQ ?
20. AB and BC do not form one straight line. Is there a point equidistant from A and B and also equidistant from B and C ?
21. Line AB intersects line CD and also line EF . Do they all lie in the same plane?
22. Can there be two planes \perp to a given plane through a given point in the given plane?
23. Plane MN is oblique to plane PQ . Can a line in one of them be \perp to the other?
24. AB is \perp to plane MN . Can a line be drawn in MN not \perp to the first line?
25. AB is \perp to plane MN . CD is \parallel to AB . Is $CD \perp MN$?
26. AB and CD are skew lines. Can a line be drawn parallel to both of them?
27. Do all the points equidistant from two intersecting planes lie in one plane?
28. Can a plane be passed through four lines all \perp to a given line at a given point?
29. If the projections of two lines on a plane are parallel, are the lines parallel?
30. If two planes are cut by a third plane, are the alternate interior dihedral angles equal?
31. A and B are points on XY . $\angle CAY = \angle DBY$. Are AC and BD parallel?

TEST X (Book Seven)

1. Can a polyhedron have only one base?
2. Are all the faces of a right prism rectangles?
3. If all the diagonals of a parallelepiped are equal, is it a right rectangular parallelepiped?
4. If the three face angles of one of the trihedral angles of a parallelepiped are equal, is it a rectangular parallelepiped?
5. If a right section of a prism is a rectangle, is the prism a rectangular prism?
6. Is the right section of an oblique prism similar to the base?
7. Is a section of a circular cylinder parallel to the base a circle?
8. Are the volumes of the two cylinders formed by revolving a rectangle about each of two adjacent sides equal?
9. May regular hexagons be used to form a regular polyhedron? —
10. May more than one type of regular polyhedron be formed by the use of equilateral triangles?
11. Can a pentahedron be regular?
12. Is the point of intersection of the diagonals of a right rectangular parallelepiped equidistant from the six faces of the parallelepiped?
13. In polyhedrons P and P' , the three faces having a common vertex of one are equal, respectively, to the three faces having a common vertex of the other. Are the polyhedrons congruent?
14. Are the diagonals of a right rectangular parallelepiped perpendicular to each other?
15. If two parallelepipeds have equal volumes, must they have equal surfaces?
16. Can a cube have the same number of square units in its area as it has cubic units in its volume?

- ✓ 17. Can a plane be tangent to a cone without containing the vertex?
18. If a cone has an altitude three times that of a cylinder, is its volume equal to that of a cylinder?
19. If a triangular pyramid and a hexagonal pyramid have equivalent bases and equal altitudes, are the areas of the pyramids equal?
20. Can a straight line have a point in common with all the lateral edges of a pyramid?
21. Is the section of a right circular cone, made by a plane which is the perpendicular bisector of the altitude, equal to half the base?
- 22. Do the areas of two similar cones of revolution have the same ratio as their altitudes?
23. If a plane is tangent to a cone, does it meet the cone in a line?
24. Are the altitudes of the lateral faces of a triangular pyramid equal?
25. Is the slant height of a cone of revolution equal to $h^2 + r^2$? (h is the altitude and r is the radius of the cone.)
26. If two circular cylinders have equal bases and equal altitudes, are they congruent?
27. If a plane is tangent to a cylinder of revolution, is it parallel to the axis?
28. Is the section of a cube cut by a plane, a parallelogram?
29. Are the sections of a cylinder made by parallel planes congruent?
30. Is the section of an oblique prism made by a plane parallel to an element a rectangle?
31. Is the section of a frustum of a cone of revolution formed by a plane through the axis an isosceles trapezoid?

Superior, 24.

Good, 22.

Passing, 18.

TEST XI (Book Eight)

1. If two spheres have equal volumes, do they have equal radii?
2. Does a chord of a sphere determine a circular section of the sphere?
3. Is the pole of a great circle of a sphere the pole of every section parallel to the plane of the given great circle?
4. Can a spherical triangle have angles of 60° , 75° , and 40° ?
5. If two circles of a sphere bisect each other, are they great circles?
6. Does the bisector of an angle of a spherical triangle bisect the side opposite the angle?
7. Can an angle of a spherical triangle contain more than 180° ?
8. Is there a sphere such that the number of square units in its area equals the number of cubic units in its volume?
9. If two plane sections of a sphere are perpendicular to each other, do they bisect each other?
10. If two non-parallel planes are each tangent to a sphere, is the angle formed by the radii to the points of contact equal to the plane angle of the dihedral angle formed by the intersection of the planes?
11. Are all the meridians of the earth perpendicular to the equator?
12. Can two spheres of unequal radii be tangent to a given plane and at the same time be tangent to each other?
13. Spherical quadrilateral $ABCD$ is equilateral. Is it equiangular?
14. If a sphere is inscribed in a cone, is a plane that is tangent to the cone tangent to the sphere?
15. If a cone is inscribed in a sphere, is a plane that is tangent to the cone tangent to the sphere?
16. Is a lune twice the area of a spherical triangle that has for one of its angles the angle of the lune?

17. If a spherical triangle is birectangular, is it isosceles?
18. Is a degree of latitude of the same length as a degree of longitude?
19. If two cones are each inscribed in equal spheres, are they equivalent?
20. If on the same sphere the angle of one lune is double that of another, does the first lune have an area double that of the second?
21. If arc PA on a sphere = 90° and arc $PB = 90^\circ$, is P the pole of a great circle through A and B ?
22. If two equal spheres are inscribed in cylinders of revolution, do the cylinders have equal volumes?
23. If PA and PB are quadrants of a sphere, is the number of spherical degrees in the area of spherical triangle PAB equal to the number of degrees in arc AB of a great circle?
24. If the volume of a sphere is double that of another, is the area of the first double that of the second?
25. If two spherical sections are perpendicular to each other, does each pass through the pole of the other?
26. If two great circles of a sphere intersect each other, do they have a common diameter?
27. If a cylinder is circumscribed about a sphere, is the lateral area of the cylinder greater than that of the sphere?
28. If two chords of a sphere are perpendicular to each other, can a great circle be passed through their end points?
29. If one angle of a spherical triangle is double another, is the side of the polar triangle opposite the first double the side of the polar triangle opposite the second?
30. If two spheres have equal areas, have they equal volumes?
31. May two places have a difference of latitude of more than 90° ?
32. Can two great circles on the same sphere be tangent to one another?

Superior, 25.

Good, 22.

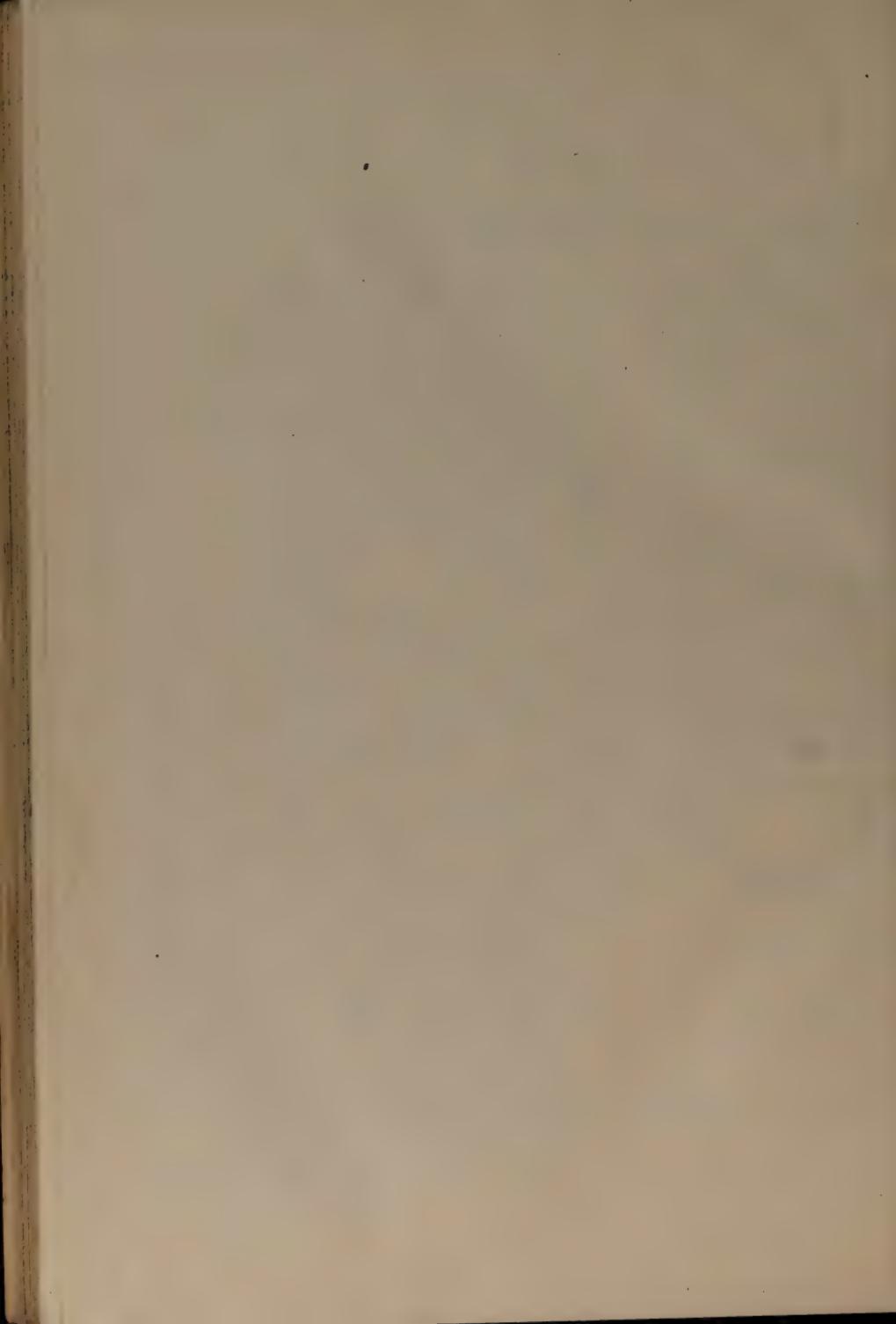
Passing, 19.

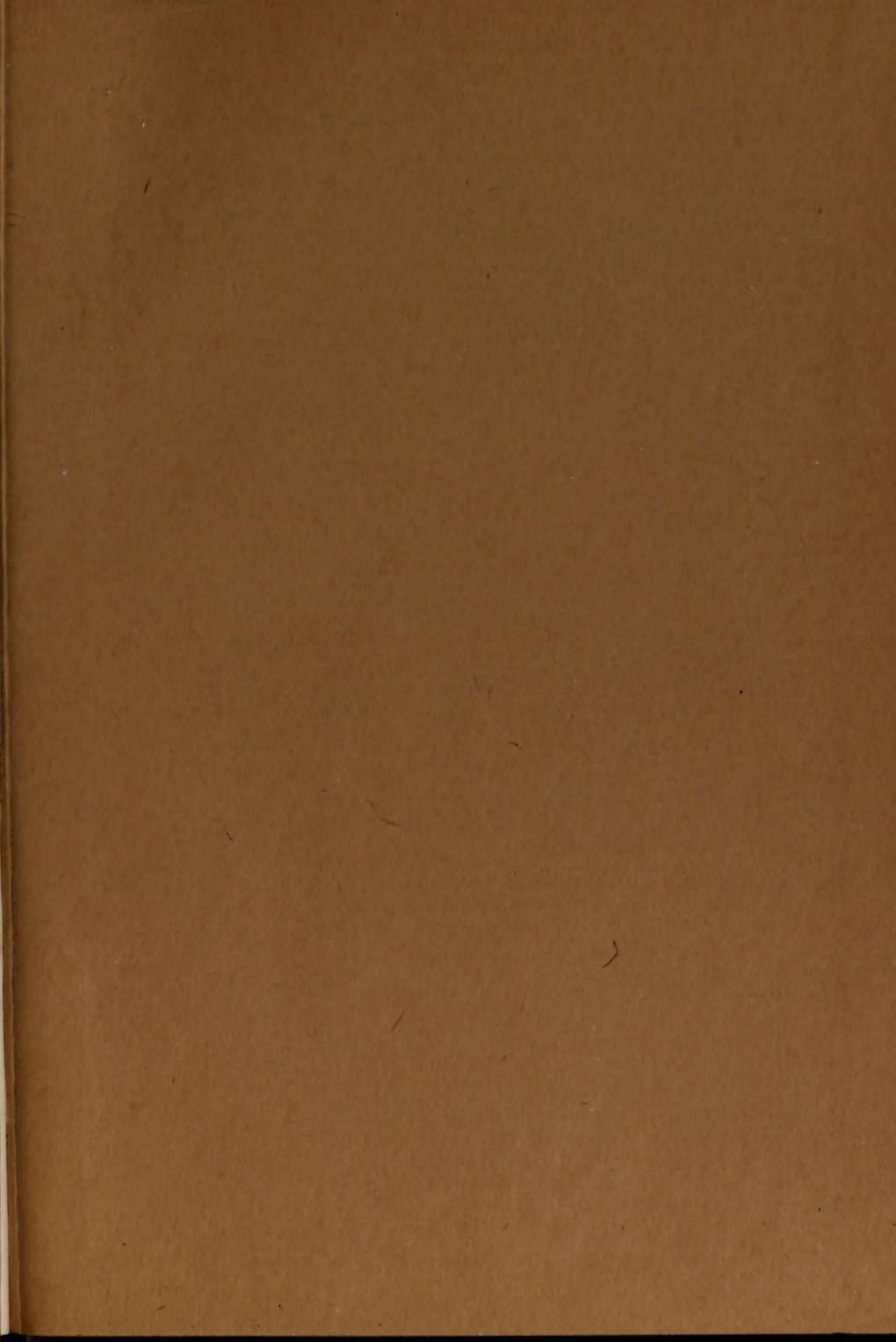
SYMBOLS AND ABBREVIATIONS

=	equals <i>or</i> is equal to	∴	therefore
≠	is not equal to	adj.	adjacent
>	is greater than	alt.	alternate, altitude
<	is less than	Ax.	Axiom
~	is similar to	Cor.	Corollary
≅	is congruent to	corr.	corresponding
⊥	perpendicular	Def.	definition
⊥s	perpendiculars	Ex.	Exercise
	parallel	ext.	exterior
s	parallels	Fig.	Figure
∠	angle	Hyp.	Hypothesis
∠s	angles	iden.	identity
△	triangle	int.	interior
△s	triangles	isos.	isosceles
□	parallelogram	opp.	opposite
□	rectangle	Post.	Postulate
○	circle	rt.	right
○s	circles	st.	straight

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AMOS & ANDY EVERY NIGHT



